Trade and the Speed of Convergence

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Abstract

This paper derives a convergence equation for a world integrated by trade. We find that factor price equalization reduces the rate of income convergence among economies with identical preferences and identical technologies. This finding hold true both in neoclassical growth models and in endogenous growth models with human capital accumulation. The integrated world model can explain low rates of convergence frequently observed in empirical studies without resorting to a large income share of capital, constraints on international borrowing, or adjustment costs in investment.

Keywords: convergence, factor price equalization, growth, integration, trade

JEL Classification: F43, O41

I. Introduction

The recent literature in growth theory has centered around the issue of convergence: whether poor countries grow faster than rich countries. In the neighboring field of trade and development, a vast literature has developed around the issue of openness and growth: Do countries with lower trade barriers grow faster? In understanding the current global capitalism, the two issues—convergence and openness and growth—are inseparable. However, the two strands of research are kept an awkward distance in their developmental process.

Research on convergence exploded following the works of Mankiw, Romer, and Weil (1992) and

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Barro and Sala-i-Martin (1992a). These studies provide a regression equation for testing the convergence hypothesis rigorously based on neoclassical growth theory. A major attraction of this approach is that the coefficients of the convergence equation can be directly linked to the structural parameters of the underlying theory, enabling researchers to assess the quantitative plausibility of the theory from estimation results.

The convergence equation is derived from the assumption that the world is composed of autarkic economies.[1] Most trade economists consider this lack of interdependence in convergence theory (Findlay,1996) disturbing. As emphasized by so many trade economists, trade in goods is effectively trade in production factors embedded in goods. Thus, economies linked by trade are likely to show the pattern of convergence very different from autarkic economies.

The main purpose of this paper is to derive a convergence equation for a world integrated by trade. We build a neoclassical model of growth in a world composed of economies with identical preferences and technologies, and examine how the equalization of factor prices through trade affects the growth rates of economies. The convergence equation for an integrated world corresponds closely to the autarkic versions obtained by Mankiw et al. (1992) and Barro and Sala-i-Martin (1992a). As theirs, our equation can be used as a regression equation for estimating the rate of convergence, and the rate of convergence

[1] Barro, Mankiw and Sala-i-Martin (1995) consider capital mobility. Capital mobility implies an infinite speed of convergence. To reduce the speed of convergence, they impose constraints on international borrowing. For the same purpose, Duczynski (2002) introduces adjustment costs in investment. Allowing trade is a different approach; factor prices are equalized without factor movements.

can be directly linked to the structural parameters of the model. However, there are some important differences: In an integrated world, (a) the growth rates of individual economies are positively affected by the growth rate of the world; (b) the rate of convergence increases as the world grows faster, and (c) the rate of convergence, under conventional parameter values, is far lower in an integrated world than in a world of autarkic economies.

To avoid overpredicting the rate of convergence in neoclassical models, Mankiw et al. (1992) and Barro et al. (1995) assumed a counterfactually large share of capital. They justified this by arguing that capital in neoclassical models should be interpreted as including both physical and human capital. However, this justification works only when physical and human capital are produced by an identical technology. Under the more plausible assumption that the production of human capital is more labor intensive, models with two capital goods generate the rate of convergence much greater than standard neoclassical models. (Ortigueira and Santos, 1997.) We thus examine how trade affects the rate of convergence in an endogenous growth model where two capital goods are created by different technologies. We show that, with trade interdependence, the rate of convergence is reduced by an order of magnitude. The integrated world model can explain low rates of convergence frequently observed in empirical studies in both neoclassical and endogenous growth models without imposing counterfactual parameter values or ad hoc restrictions on international borrowing and investment technologies.

Factor price equalization may be an assumption as extreme as that of autarkic economies. However, the exercise in this paper still seems necessary. The convergence theory is frequently applied to convergence among regions in a single country.[2] In this context, our approach makes much more sense that the alternative of assuming autarkic economies. This study also provides some valuable insights for convergence among countries that are neither autarkic nor fully integrated. Given the predominantly autarkic perspective taken by the convergence theory, this research would contribute toward balancing our view of the world.

Given the vast literature on dynamic Heckscher–Ohlin models, this paper overlaps with many existing studies. Particularly related is Ventura (1997), who examined how the substitutability between labor and capital affects convergence in an integrated world. In this paper, we restrict the elasticity of substitution between capital and labor to unity and, instead, derive the rate of convergence as a function of structural parameters and make quantitative assessments. Also related is the long tradition of research represented by Oniki and Uzawa (1965), Stiglitz (1970), and the works summarized in Findlay (1995). These studies show that the gap between a rich and a poor economy can widen with trade. Although they identified a poor economy as an economy with a higher discount rate and compared steady states, we examine a world of parametrically identical economies during transition. We show that the absolute gap

[2] See Barro and Sala-i-Martin (1992a, 1992b) for income convergence among states in the United States, and among prefectures in Japan. Barro and Sala-i-Martin (2004, chap. 11) cite a large literature on convergence among regions in a single country.

between a rich and a poor economy widens as the world grows. This paper adds to the literature on openness and growth by showing that, the widening absolute gaps are compatible with convergence in the sense used by the growth literature. The widening absolute gap points to our main result, namely, that the rate of convergence is much lower in a world integrated by trade.

This paper is organized as follows. In Section II, we examine a neoclassical model in which human capital grows at an exogenous rate. In Section III, we conduct a parallel analysis with an endogenous growth model with two capital goods, and in Section IV, we conclude.

II. Convergence in Neoclassical Growth Theory

A. General Dynamics

A closed economy produces a final good and two intermediate goods, using capital and labor. The representative household has the following objective function:

$$\begin{aligned} &\text{Max} \quad \int_0^\infty \frac{1}{1-\theta} C^{1-\theta} \exp[-(\rho-n)t]dt \\ &s.t. \quad \begin{array}{l} \dot{K} = RK + wH - C - (n+\delta)K, \\ \dot{H} = gH. \end{array} \end{aligned} \tag{1}$$

$1/\theta$ is the intertemporal rate of substitution and ρ is the subjective discount rate. The number of workers, which we denote by *L*, grows at the rate of *n*. We use *C* to denote consumption per worker, *K* to denote physical capital per worker, and *H* to denote human capital per worker. Let *R* and *w* be the rental price of physical capital and the wage rate per human capital. Physical capital depreciates at the rate of δ and human capital per worker grows at an exogenous rate of g. To ensure that utility is bounded, we assume that

$$\rho - n + (\theta - 1)\, g > 0. \tag{2}$$

We use small letters to denote quantities per effective worker. k is equal to K/H and c is equal to C/H. Then the solution for problem (1) can be expressed by the following equations:

$$\dot{c} = [\frac{1}{\theta}(R - \delta - \rho) - g]c\,, \tag{3}$$

$$\dot{k} = (R - \delta - n - g)k + w - c\,, \tag{4}$$

$$\lim_{t \to \infty}\ k\ c^{-\theta} e^{-[\rho - n + (\theta - 1)g]t} = 0. \tag{5}$$

We adopt the production structure introduced by Corden (1971) and recently used by Ventura (1997). Competitive firms produce a final good Y by bundling two intermediate goods, 1 and 2. No capital or labor is used in bundling. The bundling technology can be described as a CRS production function with diminishing marginal products: Let the unit cost of the final good be $e(p_1, p_2)$, where the arguments are the prices of intermediate good 1 and 2. We take Y as the numeraire:

$$e(p_1, p_2) = 1. \tag{6}$$

The final good is used both for consumption and for augmenting physical capital. Thus, the price of consumption and the price of investment are both equal to unity, as we implicitly assume in the household problem (1). The choice of numeraire also implies that R and w are the real rental price and wage. The real interest rate r is given by $R - \delta$.

Competitive firms produce intermediate goods 1 and 2 using the following CRS technologies with diminishing marginal products.

$$Q_1 = G_1(K_1, H_1),$$

$$Q_2 = G_2(K_2, H_2),$$

where Q_j denotes the output of intermediate good j (per worker) and $K_1 + K_2 = K$ and $H_1 + H_2 = H$. We assume that good 1 is more physical-capital-intensive than good 2. Given the prices of intermediate goods, the competitive economy allocates K and H such that GDP per worker $V(= p_1 Q_1 + p_2 Q_2)$ is maximized. Thus V, Q_1, and Q_2 can be expressed as functions of p_1, p_2, K and H. Because p_2 is a decreasing function of p_1 by equation (6), we can drop p_2 and write these functions as $V(p_1, K, H)$, $Q_1(p_1, K, H)$ and $Q_2(p_1, K, H)$. Because these functions are linearly homogenous in K and H, we can write them in per-effective-worker form:

$$v(p_1, k) \equiv V(p_1, K, H)/H,$$

$$q_1(p_1, k) \equiv Q_1(p_1, K, H)/H,$$

$$q_2(p_1, k) \equiv Q_2(p_1, K, H)/H.$$

q_1 is increasing in p_1, and q_2 is decreasing in p_1. By the Rybczynski theorem, q_1 is increasing in k, and q_2

is decreasing in k. If q_1 and q_2 are strictly positive, we can use the factor price equalization theorem to express R and w as functions of p_1 alone. Further, by the Stolper–Samuelson theorem, R is an increasing function of p_1, and w is a decreasing function of p_1.

Let y and i be final output and investment per effective worker. Then $y = c + i$. Because producing the final and two intermediate goods does not generate any profits,

$$y = v(p_1, k) = Rk + w.$$

Let e_j be the partial derivative of e with respect to p_j. For each intermediate good, output must be equal to domestic demand. Using Shepherd's lemma, this condition can be expressed as

$$\frac{q_1(p_1, k)}{q_2(p_1, k)} = \frac{e_1(p_1, p_2)y}{e_2(p_1, p_2)y}. \tag{7}$$

The Appendix shows that, using equations (6) and (7), we can express p_1 as a decreasing function of k. Thus, we can express v as a function of k alone:

$$f(k) \equiv v(p_1(k), k).$$

Because R and w are functions of p_1 and p_1 is a function of k, they are functions of k. Further, the Appendix shows that

$$R(k) = f'(k),$$

$$w(k) = f(k) - f'(k)k.$$

Because R is increasing in p_1 and p_1 is decreasing in k, R is decreasing in k. Thus, f is concave in k. The function $f(k)$ behaves just like the standard aggregate production function.

Plugging these relationships into equations (3) and (4), our economy evolves according to the following dynamic system in c and k:

$$\dot{c} = [\frac{1}{\theta}(f'(k) - \delta - \rho) - g]c \,, \tag{8}$$

$$\dot{k} = f(k) - (\delta + n + g)k - c \,. \tag{9}$$

In the following, we focus on the case in which $f(k)$ satisfies the Inada condition.[3] The economy converges to the unique steady state defined by

$$R^* = f'(k^*) = \delta + \rho + \theta g \,,$$

$$c^* = f(k^*) - (\delta + n + g)k^* .$$

We use asterisks to denote steady state values. The steady state interest rate is given by $r^* = R^* - \delta$. In short, using the aggregate production function $v(p_1(k), k)$, our two-sector model can be reduced to the one-sector model known as the Ramsey–Cass–Koopmans model.

We now consider a world composed of I countries.[4] All countries are identical to the economy previously described, except in that they have different levels of K and H. Suppose that all these countries are closed. Then, denoting country by subscript i ($i = 1, \ldots, I$), c_i and k_i would be governed by the same dynamics as described by equations (8) and (9). k_i would converge to k^* regardless of its initial level. In the long run, all countries would reach the same level of income per effective worker and would also attain the same level of income per worker if the initial levels of human capital were identical across

[3] If $f(k)/k$ has a lower bound greater than $\delta+n+g$, the economy asymtotically behaves like an AK-type endogenous growth model. The Inada condition precludes this possibility.

[4] Although we call them *countries* in a world, they can be *states*, *prefectures*, or *regions* in a single country. We prefer this interpretation because factor price equalization is more likely to hold in this environment.

countries.

Now suppose that they engage in costless trade of intermediate goods with each other. The final good is not traded. Capital and labor do not move across countries, and no international lending and borrowing occurs.[5] Let

$$\sigma_i = H_i L_i / \sum_j H_j L_j$$

be the share of country i in world effective labor, where L_i is the number of workers in country i. The share is constant, given our assumptions. We denote world variables by dropping country subscripts. For example,

$$c = \sum_i \sigma_i c_i \ ,$$

$$k = \sum_i \sigma_i k_i \ .$$

At each moment, given the value of k, we can calculate $p_1(k)$, $R(k)$ and $w(k)$. With these $R(k)$ and $w(k)$, we can also calculate physical capital intensities in the production of intermediate goods 1 and 2. Let us call them $k_1(k)$ and $k_2(k)$, respectively. We assume that

$$k_2(k) < k_i < k_1(k) \text{ for any } i. \tag{10}$$

It is well-known (Dixit and Norman, 1980) that under this condition, free trade among I countries

[5] These assumptions are not restrictive in the context of our model. Producing the final good does not cost any resources and can occur anywhere without affecting equilibrium. Factor prices are equalized across countries and households are indifferent between investing at home and lending abroad. Thus, the amount of international lending is indeterminate. Factors can move across countries (within the limit imposed by equation 10), but the levels of national income are not affected. Infinitesimal costs in cross-border movements of factors would be enough to confine them within borders.

replicates the equilibrium of the integrated economy, a hypothetical closed economy endowed with the physical capital of $\sum_i K_i$ and the labor of $\sum_i H_i L_i$. Goods and factors prices are equalized across countries at $p_1(k)$, $R(k)$, and $w(k)$. If country i is abundant in physical capital ($k_i > k$), the country exports intermediate good 1. c and k follow the path given by equations (8) and (9) and converge to the steady state of c^* and k^*. In other words, the world behaves as a single closed economy.

Because factor prices are identical across countries, consumption and capital of country i follow the following paths:

$$\dot{c}_i = [\frac{1}{\theta}(r-\rho)-g]c_i, \tag{11}$$

$$\dot{k}_i = (r-n-g)k_i + w - c_i, \tag{12}$$

where $r = f'(k) - \delta$ and $w = f(k) - f'(k)k$.

From equation (11) and the transversality condition in (5), we can obtain the following consumption function:

$$c_i = \phi(k_i + \eta), \tag{13}$$

$$\phi = 1/\int_t^\infty \exp[-\int_t^s ((r(u)-n-\frac{1}{\theta}(r(u)-\rho))du]ds,$$

$$\eta = \int_t^\infty w(s)\exp[-\int_t^s (r(u)-n-g)du]ds.$$

Equation (13) states the permanent income hypothesis. Under factor price equalization, ϕ and η are identical in all countries and it follows that $c = \phi(k+\eta)$. Using the consumption functions and equations (9) and (12), we can derive the following equation:

$$k_i - k = \exp[\int_0^t (r - n - g - \phi) ds]\ (k_i(0) - k(0))\ . \tag{14}$$

We assume throughout the paper that the world economy is below steady state and $k(0) < k^*$. Thus c and k are increasing, and r is decreasing until the world reaches steady state. Then from equation (8), we can see that $1/\theta(r - \rho) > g$ during transition. Thus,

$$\begin{aligned}\frac{1}{\phi(t)} &= \int_t^\infty \exp[-\int_t^s (r(u) - n - \frac{1}{\theta}(r(u) - \rho)) du] ds \geq \int_t^\infty \exp[-\int_t^s (r(u) - n - g) du] ds \\ &\geq \int_t^\infty \exp[-\int_t^s (r(t) - n - g) du] ds = \frac{1}{r(t) - n - g}.\end{aligned}$$

$r - n - g - \phi$ is strictly positive during transition and reaches zero in steady state. Equation (14) implies that the absolute gap in capital per effective worker between country i and the world is ever expanding during transition. In fact, the gap between any two countries is widening during transition:

$$k_i - k_j = \exp[\int_0^t (r - n - g - \phi) ds]\ (k_i(0) - k_j(0))\ .$$

This equation has a strong implication for the issue of openness and growth. Suppose that a small country i has $k_i(0) < k(0)$. If it does not trade with the world, its income will reach $y^* = R^* k^* + w^*$ in steady state. However, if it trades with the integrated world, its long-run income will be determined by $y_i^* = R^* k_i^* + w^*$, where

$$k_i^* = k^* + \exp[\int_0^\infty (r - n - g - \phi) ds]\ (k_i(0) - k(0))\ \ < k^*.$$

Thus $y_i^* < y^*$, and the country attains a lower long-run level of income under free trade. This implies that a small country whose capital per effective worker is below (above) world average will experience a lower (higher) long-run average growth rate under free trade than in an autarky.[6]

Curiously, the widening gap in capital-labor ratio does not seem to have been noted in the literature.[7] It is a direct implication of factor price equalization and permanent income hypothesis. We obtain it under general CRS technologies and general isoelastic preferences. Thus our approach also provides a generalization of Ventura (1997), who restricted technologies such that two intermediate good are produced by either capital or labor and the final good is produced by a CES production function. He also restricted the utility function to be logarithmic. A complication that arises from this generalization is that condition (10) can be violated in a finite time even if it is satisfied at time zero. If this happens, factor price equalization will fail to hold, and the dynamics as previously described is no longer applicable. In the next subsection, we show that we can avoid this problem under a popular specification of the model.

B. The Rate of Convergence under Cobb–Douglas Technology

Capital–labor ratios of countries diverge in an integrated world in (upward) transition. In the

[6] Income jumps up on the moment of opening trade due to static gains from trade. All countries gain from trade as they face the paths of prices different from autarky ones. Trade stimulates consumption of a country whose capital is below average and this additional consumption reduces its long-run level of income.

[7] A study that anticipated this result is Atkeson and Kehoe (2000), who found that the moment a small country opens trade to a big country in steady state, it stops growing. Note that our result covers this stagnation as a special case where $r - n - g - \phi = 0$.

literature, convergence is usually defined in terms of the gap in logarithmic income $\left|\log y_i - \log y_j\right|$, which is approximately equal to the gap in relative income $\left|y_i / y_j - 1\right|$. In this subsection, we show that the widening absolute gap is consistent with convergence in relative income and only implies a lower rate of convergence under Cobb–Douglas technology and conventional parameter values.

The pattern of convergence in a world of autarkic economies is well understood for the Cobb–Douglas case. By log-linearizing the system composed of equations (8) and (9) with the assumption that $f(k) = Ak^{\alpha}$, Barro and Sala-i-Martin (1992a) derive the following equation of convergence for autarkic economies:

$$\frac{1}{T}\log\frac{y_i(T)}{y_i(0)} = \frac{1}{T}(1-\exp[-\lambda_c T])\log\frac{y_i{}^*}{y_i(0)}, \tag{15}$$

where

$$\lambda_c = \frac{1}{2}[\sqrt{(R^*-\delta-n-g)^2 + 4(1-\alpha)\frac{R^*}{\theta}(\frac{R^*}{\alpha}-\delta-n-g)} - (R^*-\delta-n-g)],$$

$$R^* = \delta + \rho + \theta\ g\ .$$

λ_c is the rate of convergence and is always positive. It measures the rate at which the gap between current income (log y_i) and steady state income ($\log y_i^*$) shrinks in a single, closed economy. If countries in the world are parametrically identical and they all are in autarky,

$$\log\frac{y_i(T)}{y_j(T)} = \exp[-\lambda_c T]\log\frac{y_i(0)}{y_j(0)}.$$

Thus λ_c also determines the rate at which the relative gap in income between two countries shrinks.

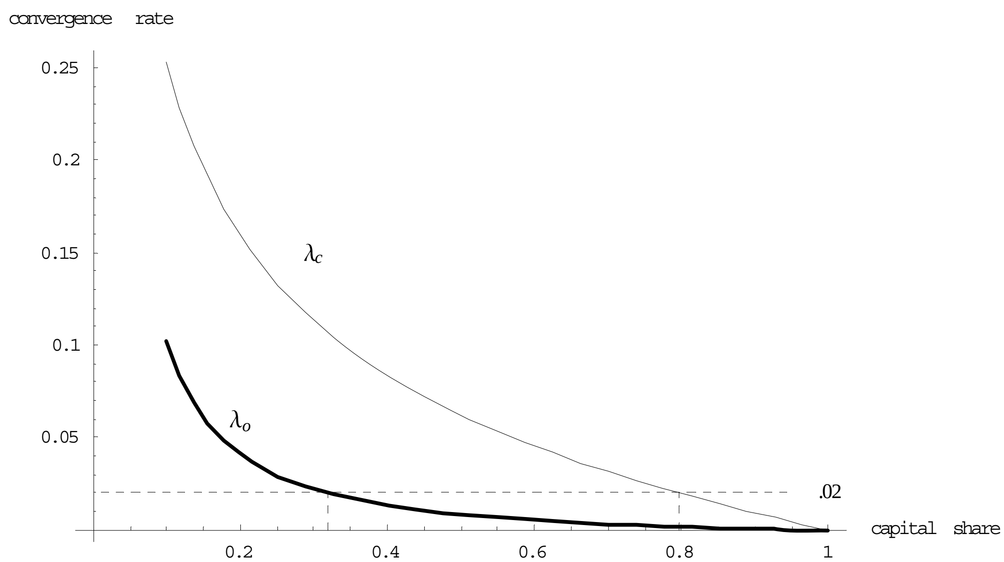

Figure 1. Rate of Convergence in the Neoclassical Model

The standard method for estimating λ_c from data is to run a cross-country (or cross-region) regression of equation (15). Many empirical studies estimate λ_c to be around the value of .02.[8]

Can neoclassical growth theory predict this slow rate of convergence under plausible parameter values? We take the following values as the benchmark case:

$$\theta = 1,\ \rho = 0.02,\ \delta = 0.05,\ n = 0.01,\ g = 0.02.$$

It is well-known that λ_c is sensitive to α, capital share of income. The upper line in Figure 1 traces the

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[8] In a standard cross-country regression, y_i^* is allowed to vary among countries, implying *conditional convergence*. In a cross-regional regression, it is frequently assumed that regions approach the same steady state. So many growth regressions report λ_c close to 0.02 that some regard it as an empirical law. See Barro and Sala-I-Martin (2004, chaps. 11–12). However, estimation methods are controversial. Caselli, Esquivel, and Lefort (1996), Islam (1995), and many others use variants of fixed-effects panel estimation and find that λ_c is far greater than 0.02. The panel estimations eliminate omitted variable biases, but also introduce upward bias due to measurement errors. See Durlauf, Johnson, and Temple (2005) and Hauk and Wacziarg (2004) for a recent evaluation of this controversy.

value of λ_c as α changes from zero to 1. At a conventional value of α, which is around 0.30, λ_c is well above 0.10. To reduce it to 0.02, α should be as high as 0.80. For this reason, Mankiw et at. (1992) suggested that α be interpreted as the share of broadly defined capital, including both human and physical capital.

Let us derive the equation of convergence for an integrated economy. We manipulate equation (14) to obtain the relative gap in capital between country *i* and the world:

$$\frac{k_i}{k} - 1 = \exp\left[\int_0^t (r - n - g - \phi - g_k) ds\right] \left(\frac{k_i(0)}{k(0)} - 1\right) . \tag{16}$$

We use g_x to denote the growth rate of *x*. Using the fact that $y_i = Rk_i + w$ and equation (16), we turn this into an equation for the relative gap in income:

$$\frac{y_i}{y} - 1 = \exp\left[\int_0^t (r - n - g - \phi + g_R - g_y) ds\right] \left(\frac{y_i(0)}{y(0)} - 1\right) . \tag{17}$$

The Appendix shows that if the production functions for the final good and the two intermediate goods are of Cobb–Douglas form, the aggregate production function $f(k)$ also takes a Cobb–Douglas form: $f(k) = A\ k^{\alpha}$. A is a positive constant that depends on the parameters of the three Cobb–Douglas production functions. α is given by the weighted average of capital shares in the production of two intermediate goods, the weights given by the shares of intermediate goods in the final good. This result is convenient for us because we can interpret the aggregate Cobb–Douglas production function of Barro and Sala-i-Martin (1992a) as the one derived from our two-sector economy. We can make a direct comparison between their results and ours.

With the Cobb–Douglas assumption, Rk/y is constant at α. Thus, $g_R + g_k = g_y$. By equation (17),

$$\frac{y_i(T)}{y(T)} - 1 = \exp[-\lambda_o T](\frac{y_i(0)}{y(0)} - 1), \tag{18}$$

$$\lambda_o = \frac{1}{T}\int_0^T [g_k - (r - n - g - \phi)]dt \,. \tag{19}$$

Both g_k and $r - n - g - \phi$ are positive during transition, and the sign of λ_o, in general, is indeterminate. Using the Taylor approximation that $\log y_i / y \approx y_i / y - 1$ and rearranging terms, equation (18) can be written as:

$$\frac{1}{T}\log\frac{y_i(T)}{y_i(0)} \approx \frac{1}{T}\log\frac{y(T)}{y(0)} + \frac{1}{T}(1 - \exp[-\lambda_o T])\log\frac{y(0)}{y_i(0)} \,. \tag{20}$$

Compared with equation (15), we notice two differences. First, the growth rate of country *i* now depends on the world growth rate, $1/T \log y(T)/y(0)$. Second, we have $y(0)$ in place of y_i*.[9] In a cross-country or a cross-region regression, in which the growth rate and the initial income of the world can be treated as constants, we can estimate λ_o using the same regression equation used in estimating λ_c. Thus, in standard growth regressions using data for economies connected by trade, researchers may be obtaining estimates for λ_o, not λ_c. This misappropriation would be far more likely to occur when a regression is over regions in a single country.

What is the sign of λ_o? The Appendix shows that $\lambda_o > 0$ if and only if the saving rate of the integrated world decreases as it approaches steady state from below. The condition holds if and only if

[9] We show below an additional difference: the rate of convergence is an increasing function of the world growth rate. These differences can be tested in a panel regression.

$$\frac{R^*}{\theta} > \alpha(\delta + n + g). \tag{21}$$

If $\theta = 1$, condition (21) always holds. The relative income gap shrinks during transition and we have convergence. If we assume that $\theta = 2$, keeping the other parameters at the benchmark levels, the condition holds for $\alpha < .69$. With $\theta = 3$, the condition holds for $\alpha < .54$. Interestingly, if θ is greater than 2 (as many believe), at the high value of α suggested by Mankiw et al. (1992), relative income diverges.

To obtain a numerical value of λ_o, we first examine the case of $\theta = 1$. In this case, $\phi = \rho - n$ and $g_k - (r - n - g - \phi) = g_k - g_c$. Because $y = Ak^a$, $g_k = \frac{1}{\alpha} g_y$. The Appendix shows that $g_c = \frac{1-\alpha}{\lambda_c} \frac{R^*}{\theta} \frac{1}{\alpha} g_y$ on the convergent path of the log-linearized system. Plugging these equations into equation (19), we find

$$\lambda_o = \frac{1}{\alpha}(1 - \frac{1-\alpha}{\lambda_c} R^*) \ x, \tag{22}$$

where

$$x = \frac{1}{T} \log \frac{y(T)}{y(0)}.$$

The rate of convergence is increasing in x, the growth rate of world income per effective worker. If the world initially is in steady state, the world stops growing and the rate of convergence becomes zero.

λ_o under various parameter values can be calculated from equation (22), once we know the value of x, which can be obtained from data. For convergence among U.S. states, we can use the growth rate of U.S. GDP per effective worker. Using the numbers and the method used by Jones (2002), between 1960 and 1990, U.S. GDP per work hour increased at the rate of .0186 per year, while an estimate of human capital increased at the rate of .0065. Thus, x is given by 0.012. For convergence among countries, we use the Penn World Table (Heston, Summers, and Aten, 2002) and the education data from Barro and Lee (2001). Between 1960 and 1990, the world income grew at the rate of 0.0385, and effective workers in the world grew at the rate of 0.023. In this case, x is given by .0155. In the following, we use 0.015 as the value of x. The lower line of Figure 1 traces the value of λ_o as α changes. The rate of convergence in the integrated world is much lower than that in the world of autarkic economies. To have the rate of convergence equal to 0.02, we need $\alpha = 0.32$, which is close to the shares of physical capital observed in most countries.

For $\theta \neq 1$, we cannot obtain a simple formula like equation (22) as ϕ depends on the future path of the interest rate. We numerically simulate the model to obtain λ_o. We obtain the convergent paths of k and r from the standard Ramsey model and directly calculate the value of the integral in equation (19) on these paths. We select the initial value of k such that the average annual growth rate of the world for 30 years is equal to 0.015. The results are reported in Table 1.[10] Note that in all cases the rate of convergence is much smaller in an integrated world than in a world of autarkic economies. Increasing the value of x reduces the difference, but unless we adopt an absurdly large value of x, the order does not change. The rate of convergence gests somewhat lower with $\theta > 1$. With $\theta = 2$, $\lambda_o = 0.02$ for $\alpha = 0.025$; and with $\theta = 3$, $\lambda_o = 0.02$ for $\alpha = 0.022$. As we expected, λ_o becomes negative for $\alpha > 0.69$ with $\theta = 2$, and for $\alpha > 0.54$ with $\theta = 3$. However, these negative values are so small in absolute value that it is virtually zero. These

[10] We do not report the result for $\theta = 1$ as it is very close to the log-linear approximation in Figure 1.

Table 1--Simulation Results for the Neoclassical Model

	$\theta = 2$		$\theta = 3$	
	λ_c	λ_o	λ_c	λ_o
0.1	0.265	0.098	0.249	0.085
0.2	0.160	0.032	0.147	0.024
0.3	0.109	0.014	0.099	0.009
0.4	0.078	0.006	0.070	0.003
0.5	0.056	0.003	0.049	0.001
0.6	0.040	0.001	0.034	–0.000
0.7	0.027	–0.000	0.023	–0.001
0.8	0.016	–0.000	0.013	–0.000
0.9	0.007	–0.000	0.006	–0.000

results are not sensitive to changes in the value of ρ, δ, n, or g.

We go back to the question regarding whether factor price equalization condition (10) will be violated during transition. We can easily show that under the Cobb−Douglas assumption, k_2/k and k_1/k are constant. Thus, if the rate of convergence is positive, $|k_i / k - 1|$ decreases and the condition will never be violated if it is satisfied at time zero. The simulation results show that the condition is likely to be maintained even with a negative rate of convergence as it is so close to zero.

III. Convergence in Two-Capital Endogenous Growth Models

A. General Dynamics

The analysis of this section parallels with Section II except that we now allow human capital to be endogenously determined. A representative consumer maximizes the following objective.

$$\text{Max} \int_0^\infty \frac{1}{1-\theta} C^{1-\theta} \exp[-(\rho - n)t]dt$$

$$\text{s.t.} \quad \dot{K} = RK + wH - C - (n+\delta)K - \pi Z,$$

$$\dot{H} = g(\frac{Z}{H})H.$$

In the $\dot{K}$ equation, we enter the term πZ, where π is the unit cost of education and Z is the amount of education per worker. g now is an increasing function of Z/H. When there are no adjustment costs in human capital accumulation, g is linear in z:

$$g(z) = z - \delta_H, \tag{23}$$

where $z = Z/H$ and δ_H is the depreciation rate of human capital. Alternatively, we can introduce adjustment costs in human capital accumulation by assuming that $g(z)$ is strictly concave. To analyze this case, we use a specific form:

$$g(z) = z^* + \frac{1}{a}(\sqrt{1+2a(z-z^*)} - 1) - \delta_H. \tag{24}$$

z^* is the steady state value of z, which is defined later. a is a parameter that captures the size of adjustment costs. As a goes to zero, equation (24) approaches equation (23).

For a closed economy, we can solve the model in the following way. We now have three production sectors: two sectors producing intermediate goods and one sector producing education. A three-sector growth model is hard to handle algebraically, but we can reduce it to a two-sector model. Let K_Y and H_Y denote physical and human capital allocated to the production of intermediate goods, where $K_Y = K_1 + K_2$

and $H_Y = H_1 + H_2$ (subscripts 1 and 2 denote two intermediate goods). Applying the result in Section II, the output of Y can be expressed as a single CRS production function $F(K_Y, H_Y)$. If the production functions for the intermediate goods and the final good are Cobb–Douglas functions, F also is a Cobb–Douglas function. The remaining resources K_Z (= $K - K_Y$) and H_Z (= $H - H_Y$) go into the education sector to produce Z, where $Z = G(K_Z, H_Z)$, a CRS production function with diminishing marginal products. We assume that Y is more physical-capital intensive than Z. We can apply the theorems of the Heckscher–Ohlin model to this (upper-level) two-sector structure composed of sectors Y and Z. By the factor price equalization theorem, R and w can be expressed as functions of π. By the Stolper–Samuelson theorem, $R(\pi)$ is decreasing in π and $w(\pi)$ is increasing in π. The economy allocates K and H between Y and Z such that $Y + \pi Z$ is maximized. Thus, we can express the outputs of Y and Z as functions of π, K, and H. In per-effective-worker form, $y = y(\pi, k)$ and $z = z(\pi, k)$. y is decreasing in π, and z is increasing in π. By the Rybczynski theorem, y is increasing in k, and z is decreasing in k.

The Appendix shows that the following equations characterize the optimal path.

$$\dot{\pi} = \frac{1}{1 - \frac{g''(z)}{g'(z)} z_\pi \pi}[(r - n - g(z))\pi - (w - \pi\ z)g'(z) + \frac{g''(z)}{g'(z)} z_k \pi\ (y - c - (\delta + n + g(z))k)], \tag{25}$$

$$\dot{c} = (\frac{1}{\theta}(r - \rho) - g(z))c\,, \tag{26}$$

$$\dot{k} = y - c - (\delta + n + g(z))k \tag{27}$$

Plugging in the functions $r(\pi)$ (=$R(\pi)-\delta$), $w(\pi)$, $y(\pi, k)$, and $z(\pi, k)$, the three equations (25–27) constitute a three-dimensional dynamic system in π, c, and k, where π and c are jumping variables and k is a state variable. In steady state, $\dot{\pi} = \dot{c} = \dot{k} = 0$. This implies that

$$\frac{w(\pi^*)}{\pi^*} = r(\pi^*) - n + \delta_H \,, \tag{28}$$

$$\frac{1}{\theta}(r(\pi^*) - \rho) = g(z(\pi^*, k^*)) \,, \tag{29}$$

$$c^* = y(\pi^*, k^*) - (\delta + n + g(z(\pi^*, k^*)) \;\; k^* \,. \tag{30}$$

We define $z^* \equiv z(\pi^*, k^*)$ and $g^* \equiv g(z^*)$.

When there are no adjustment costs, equation (25) is simplified to

$$\dot{\pi} = (r - n + \delta_H)\pi - w \,. \tag{31}$$

The system composed of equations (31), (26), and (27) has been studied by other authors, including Bond, Wang, and Yip (1996).[11] The steady state defined by equations (28) through (30) exists and is unique and the system is saddle-point stable. Along the stable manifold, π and c are increasing functions of k. We build on this result.

Suppose that all countries in the world are identical except in the initial endowments of K and H. If these countries are closed, they will independently reach an identical steady state (π^*, c^*, k^*). Thus GDP per effective worker, $y(\pi, k) + \pi z(\pi, k)$, will be equalized across countries in steady state.[12]

[11] For the case where Z is produced using only human capital, which is often called the Uzawa–Lucas model, Caballé and Santos (1993) derived analytical solutions. Mulligan and Sala-i-Martin (1993) provided numerical simulation results for a broad class of two-sector endogenous growth models, with or without adjustment costs, under the assumption that goods are produced by Cobb–Douglas technologies.

[12] However, GDP per worker will not be equalized, even when $H_i(0)$ are identical in all countries. Different values of $K_i(0)$ imply different paths of human capital.

What will happen if countries trade two intermediate goods? We assume that Z is not tradable. In this case also, if countries are not too much different from each other in factor proportion, trade replicates the equilibrium of an integrated economy. Factor prices R and w are equalized across countries as the prices of two intermediate goods are equalized through trade. Importantly, π also is equalized across countries, even though Z is not tradable. As countries have identical factor prices and have an identical CRS production function for Z, the unit cost of Z is identical in all countries. Again, the world behaves like a single, closed economy. Under our assumption that the initial value of k is less than k^*, π, c, and k will increase over time toward their steady state values. According to the Stolper–Samuelson theorem, r will decrease and w will increase over time.

However, without adjustment costs in investment, an indeterminacy problem arises at the level of individual countries. In an open economy where the outputs of two intermediate goods do not have to be equal to their domestic demands, for a given value of π and k_i, there exist infinitely many values of y_i and z_i that clear factor markets.[13] The Appendix explains how we can break this indeterminacy by introducing small adjustment costs in human capital accumulation along the line of the q theory of investment. With adjustments costs, the demand for z_i is determined by an increasing function of q, the shadow price of human capital divided by π. As factor prices are equalized, q also is equalized across countries. Thus, z_i is

[13] This indeterminacy is a well-known problem in international trade that arises in a model with three goods and two factors. See Dixit and Norman (1980).

identical in all countries and $\dot{H}_i / H_i = g(z(\pi, k))$ for every i.[14]

Consumption in each country is determined by $c_i = \phi(k_i + \eta)$. The definition of ϕ is the same as before, but we change the definition of η to

$$\eta \equiv \int_t^\infty (w(s) - \pi\ z(s)) \exp[-\int_t^s (r(u) - n - g(z(u))du]ds .$$

ϕ and η are identical in all countries because w, r, and z are identical. Plugging the consumption function into equation (27) and using the relationship that $y + \pi z = w + r k$, we derive the following equation:

$$k_i - k = \exp[\int_0^t (r - n - g(z) - \phi)ds](k_i(0) - k(0)) . \tag{32}$$

Note that the only difference form the neoclassical case is that g now depends on z.

The behavior of the absolute gap in k depends on the sign of $r - n - g(z) - \phi$. c is increasing in transition and thus $\frac{1}{\theta}(r - \rho) > g(z)$. r is decreasing in transition. If z is increasing (Mulligan and Sala-i-Martin (1993) show that it is very likely to be increasing in most two sector models), using the same method as we use in Section II, we can show that $r(t) - n - g(z(t)) > \phi(t)$ for all t. Equation (32) implies that the absolute gap $k_i - k$ is ever widening during transition, as in the neoclassical model.

Again, a small country whose capital–labor ratio is below world average will attain a higher long-run level of income per effective worker in autarky than under free trade. However, the country may end up with a higher level of income per worker under free trade. We can show that a country with $k_i(0) < k(0)$ attains a higher steady state level of H_i under free trade.

[14] Because z_i is identical in all countries, it must be equal to the world average z. Since the world is a closed economy, the function $z(\pi, k)$ is well-defined.

B. The Rate of Convergence under Cobb–Douglas Technology

We now impose the assumption that production functions for the final good, two intermediate goods, and education are of Cobb–Douglas form. We introduce adjustment costs in human capital accumulation, but we keep them infinitesimal. As *a* in equation (24) goes toward zero, the path of the world economy approaches the one without adjustment costs. However, adjustment costs—however small—prevent indeterminacy. By examining this limit economy, we can extract the pure effect of trade on the speed of convergence and make a controlled comparison with the well-known results derived from autarkic economies without adjustment costs.[15] .

We start with a world of autarkic economies. We log-linearize the system composed of equations (31), (26), and (27). In the Appendix, carefully employing the mathematical properties of a 2 × 2 model, we derive explicit solutions for the three eigenvalues of the system:

$$\mu_1 = R'(\pi^*)(k^* + \pi^*) + R^* - \delta - n - g^* < 0 ,$$

$$\mu_2 = -R'(\pi^*)(k^* + \pi^*) > 0 ,$$

$$\mu_3 = R^* - \delta - n - g^* > 0 .$$

The dynamic system is saddle-point stable. Let $\overline{\lambda}_c = -\mu_1$ > 0. On the convergent path,

$$\log k - \log k^* = \exp[-\overline{\lambda}_c t](\log k(0) - \log k^*) , \tag{33}$$

$$\log \pi - \log \pi^* = m_\pi (\log k - \log k^*) , \tag{34}$$

[15] The effects of adjustment costs have been studied in other papers. See Duczynski (2002) for an analysis in neoclassical models, and Ortigueira and Santos (1997) for an analysis in endogenous growth models.

$$\log c - \log c^* = m_c(\log k - \log k^*)\,, \tag{35}$$

$$m_\pi = \frac{-\frac{k^*}{\pi^*}(2R'(k^*+\pi^*) + R^* - \delta - n - g^*)}{z_\pi(k^*+\pi^*) + \frac{1}{\theta}(R^* - \delta - n - g^*)} > 0,$$

$$m_c = \frac{k^*(z_\pi - \frac{2}{\theta}R')}{z_\pi(k^*+\pi^*) + \frac{1}{\theta}(R^* - \delta - n - g^*)} > 0\,.$$

We denote GDP per effective worker by $v(\pi, k) = y(\pi, k) + \pi z(\pi, k)$. Using a log-linear approximation around the steady state,

$$\begin{aligned}\log v(T) - \log v^* &\approx \frac{v_\pi \pi^*}{v^*}(\log \pi(T) - \log \pi^*) + \frac{v_k k^*}{v^*}(\log k(T) - \log k^*),\\ &= \exp[-\bar{\lambda}_c T]\left(\frac{v_\pi \pi^*}{v^*}(\log \pi(0) - \log \pi^*) + \frac{v_k k^*}{v^*}(\log k(0) - \log k^*)\right),\\ &\approx \exp[-\bar{\lambda}_c T](\log v(0) - \log v^*).\end{aligned}$$

Rearranging the terms,

$$\frac{1}{T}\log\frac{v(T)}{v(0)} = \frac{1}{T}(1 - \exp[-\bar{\lambda}_c T])\log\frac{v^*}{v(0)}\,. \tag{36}$$

Note that we have exactly the same form of the equation as we obtained for the neoclassical model. Because $\bar{\lambda}_c > 0$, the endogenous growth model also implies conditional convergence.[16]

Using the eigenvalues of the dynamic system, the Appendix obtains an explicit expression for the rate of convergence for an autarkic economy:

[16] Despite the suggestions of Mulligan and Sala-i-Martin (1993) and Ortigueira and Santos (1997), many researchers still believe that endogenous growth models are consistent with conditional convergence only in some special cases. As we demonstrate here, both the neoclassical and the two-capital endogenous growth model imply exactly the same form of convergence.

$$\bar{\lambda}_c = \frac{1-(\alpha-\beta)}{\alpha-\beta} R^* + \frac{\beta}{\alpha-\beta}(\delta_H - \delta - n)\,. \qquad (37)$$

α is capital share in the production of the final good, and β is capital share in education. Ortigueira and Santos (1997) derived the following formula from the Uzawa–Lucas model ($\beta = 0$):

$$\bar{\lambda}_c = \frac{1-\alpha}{\alpha} R^*$$

This is a special case of equation (37). Ortigueira and Santos found that, under widely used parameter values, this rate of convergence is much higher than the one in neoclassical growth models. Equation (37) shows that the problem worsens with β strictly positive. Here, we cannot broaden the definition of capital to avoid the problem of overpredicting the rate of convergence. Human capital accumulation is already incorporated in the model.

Now we derive the counterpart equation for an integrated world. Using equation (32) and the fact that $v_i - v = R(k_i - k)$, we derive the following equation:

$$\frac{v_i(T)}{v(T)} - 1 = \exp[-\bar{\lambda}_o\ T](\frac{v_i(0)}{v(0)} - 1)\,,$$

where

$$\bar{\lambda}_o \equiv \frac{1}{T}\int_0^T [g_v - g_R - (r - n - g(z) - \phi)]dt\,.$$

If we use the approximation that $v_i / v - 1 \approx \log v_i / v$ and rearrange the terms,

$$\frac{1}{T}\log\frac{v_i(T)}{v_i(0)} = \frac{1}{T}\log\frac{v(T)}{v(0)} + \frac{1}{T}(1-\exp[-\bar{\lambda}_o T])\log\frac{v(0)}{v_i(0)}\,. \qquad (38)$$

Again, we obtain the same form of equation as in the neoclassical case.

To find how trade affects the rate of convergence, we start with the case where $\theta = 1$. In this case, $\phi = \rho - n$ and $r - n - g - \phi = g_c$. The Appendix shows that

$$\bar{\lambda}_o = \frac{(\frac{\pi^* z^*}{v^*} - \frac{R'\pi}{R})m_\pi + \frac{R^* k^*}{v^*} - m_c}{\frac{\pi^* z^*}{v^*} m_\pi + \frac{R^* k^*}{v^*}} \; x\,, \tag{39}$$

where $x = \frac{1}{T}\log\frac{v(T)}{v(0)}$. Again the rate of convergence is increasing in x. Using the Cobb–Douglas assumption, all variables except x in equation (39) can be expressed as functions of parameters. The Appendix shows that in the Uzawa–Lucas model where $\beta = 0$,

$$\frac{\pi^* z^*}{v^*} = \frac{(1-\alpha)(g^* + \delta_H)}{R^* - \delta - n - g^* + (1-\alpha)\ (g^* + \delta_H)}\,,$$

$$\frac{R^* k^*}{v^*} = \frac{\alpha(R^* - \delta - n - g^*)}{R^* - \delta - n - g^* + (1-\alpha)\ (g^* + \delta_H)}\,,$$

$$\frac{R'\pi}{R} = -\frac{1-\alpha}{\alpha}\,,$$

$$m_\pi = \frac{\alpha[R^* - \alpha(\delta + n + g^*)] + (1-\alpha)\alpha R^*}{R^* - \alpha(\delta + n + g^*) + (1-\alpha)\alpha R^*}\,,$$

$$m_c = \frac{\alpha(R^* - \delta - n - g^*) + 2(1-\alpha)\alpha R^*}{R^* - \alpha(\delta + n + g^*) + (1-\alpha)\alpha R^*}\,.$$

Using these functions, we can calculate numerical values of $\bar{\lambda}_o$ for the bench mark parameter values. Lacking information about the depreciation rate of human capital, we assume that it is equal to that of physical capital. z^* can be obtained from the relationship that $z^* = g^* + \delta_H$. Again, we assume that $x = 0.015$. Figure 2 traces the value of $\bar{\lambda}_c$ and $\bar{\lambda}_o$ as α changes. The rate of convergence is much lower in an integrated world. In a world of autarkic economies, α should be higher than 0.80 to have the rate of

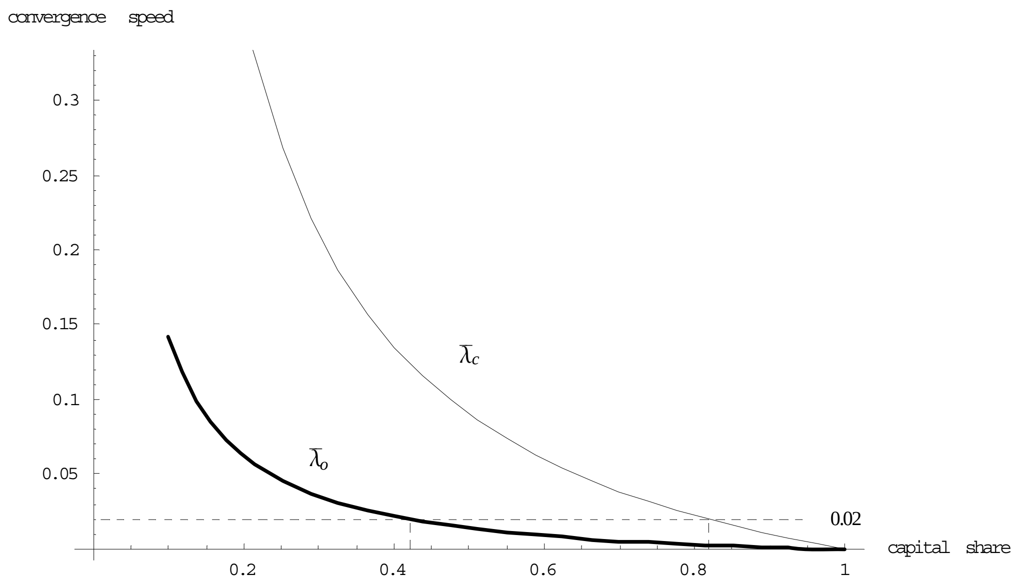

Figure 2. Rate of Convergence in the Uzawa–Lucas Model

$\theta = 1$, $\rho = 0.02$, $\delta = \delta_H = 0.05$, $n = 0.01$, $g^* = 0.02$, $R^* = \delta + \rho + \theta g^*$, $z^* = g^* + \delta_H$.

convergence equal to 0.02. To have the rate of convergence equal to 0.02 in an integrated world, we need $\alpha = 0.42$. (α is capital share in the final good sector. Because education is more labor intensive, the capital share of the entire economy is lower than α.) In the same way, we can examine a general endogenous growth model where physical capital is used in education. The formula for $\bar{\lambda}_o$ gests much more complicated and we report it only in the Appendix. We use the value of $\beta = 0.10$ and repeat the same analysis as previously described. The results are reported in Figure 3. With β greater than zero, the rate of convergence gets higher both in a world of autarkic economies and in an integrated world. In the autarkic case, α should be as high as 0.87 to have the rate of convergence equal to 0.02. In an integrated world, α

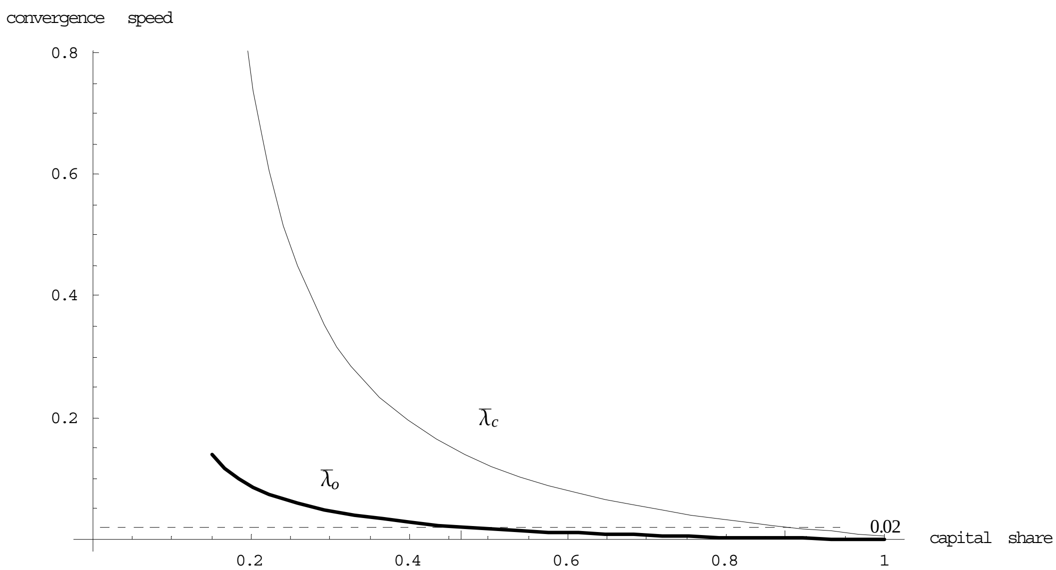

Figure 3. Rate of Convergence in the General Endogenous Growth Model

$\beta = .1$, $\theta = 1$, $\rho = 0.02$, $\delta = \delta_H = 0.05$, $n = 0.01$, $g^* = 0.02$, $R^* = \delta + \rho + \theta g^*$, $z^* = g^* + \delta_H$.

= 0.46 generates the rate of convergence equal to 0.02. To analyze the case where $\theta > 1$, we use the simulation method. Table 2 summarizes the results for $\alpha = 0.4$. In an autarkic economy, the rate of convergence slightly increases as θ increases.[17] However, the rate of convergence is insensitive to θ in an integrated world. This is also true for other values of α. Thus the graphic pattern in Figure 3 also holds with θ greater than unity.

[17] As Ortigueira and Santos (1997) showed, R^* and thus $\bar{\lambda}_c$ should be independent of preferences. Here $\bar{\lambda}_c$ increases with θ because of our calibration method. As we raise θ, we raise R^* at the same time, keeping the steady state relationship $R^* = \delta + \rho + \theta g^*$ with the constant values of δ, ρ, and g^*.

Table 2--Simulation Results for the Endogenous Growth Model: $\alpha = 0.4$

	$\theta = 2$		$\theta = 3$	
β	$\bar{\lambda}_c$	$\bar{\lambda}_o$	$\bar{\lambda}_c$	$\bar{\lambda}_o$
0.0	0.177	0.023	0.208	0.023
0.1	0.264	0.027	0.311	0.028
0.2	0.484	0.034	0.517	0.035

IV. Conclusion

We find that a strong tendency exists for absolute gaps in income between economies to widen in an integrated world. However, widening gaps in absolute income are compatible with declining gaps in relative income, and thus the evidence for convergence in the growth literature. In fact, under conventional parameter values, these widening absolute gaps work toward reducing the rate of convergence down to empirically observed values. These findings hold true both in neoclassical models and in two-capital endogenous growth models.

This paper provides further theoretical background for the studies such as Rodriguez and Rodrik (2000) and Slaughter (2001), who argued that no robust evidence exists that trade speeds up convergence. However, we do not argue that trade would actually slow down convergence in the real world: The models that we use are highly stylized and fail to reflect many important aspects. The most serious omission is technological progress, which would be at least as important as capital accumulation in determining world distribution of income. However, we anticipate that our convergence equation for an integrated world may aid empirical researchers in assessing the role of capital accumulation and technological progress in the evolution of world income distribution.

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Mathematical Appendix

***1**. In the neoclassical two-sector closed economy, p_1 is a decreasing function of k, and*

$$R(k) = f'(k),$$

$$w(k) = f(k) - f'(k)\, k.$$

Proof)

The left-hand side of equation (7), the relative supply, is increasing in p_1. The right-hand side, the relative demand, is decreasing in p_1 because p_2 is a decreasing function of p_1. We assume that the relative demand goes to infinity as p_1 goes to zero and goes to zero as p_1 goes to infinity. (This assumption is satisfied by a CES technology.) Then the value of p_1 satisfying equation (7) exists and is unique for any level of k. By the Rybczynski theorem, the relative supply is increasing in k. This implies that the equilibrium value of p_1 is decreasing in k.

The competitive two-sector economy allocates resources as if it solves the following problem.

$$v(p_1, p_2, k) \equiv \underset{K_1, K_2, H_1, H_2}{Max}\; p_1 G_1(K_1, H_1) + p_2 G_2(K_2, H_2)$$

$$\text{s.t.} \quad K_1 + K_2 \le k,$$

$$H_1 + H_2 \le 1.$$

Using the envelope theorem, we can show that

$$\frac{\partial v}{\partial p_1} = q_1, \;\; \frac{\partial v}{\partial p_2} = q_2, \;\; \frac{\partial v}{\partial k} = R.$$

From equation (6), $e_1 dp_1 + e_2 dp_2 = 0$. Thus

$$dv = (q_1 dp_1 + q_2 dp_2) + Rdk = (\frac{q_1}{q_2} - \frac{e_1}{e_2}) q_2 dp_1 + Rdk \,.$$

In autarky equilibrium, the term in the parenthesis is zero. Thus $dv/dk = f'(k) = R$. Since $f(k) = Rk + w$, $w = f(k) - f'(k)\, k$.

2. *In the neoclassical two-sector closed economy, the production functions for the final good and the two intermediate goods are given by the following Cobb-Douglas functions.*

$$Y = A_f Q_1^{\ m} Q_2^{\ 1-m} \,,$$

$$Q_j = A_j K_j^{\ \alpha_j} (H_j)^{1-\alpha_j} \ (j = 1, 2).$$

Then

$$f(k) = Ak^{\alpha} \,,$$

where A is a positive constant and $\alpha = m\ \alpha_1 + (1-m)\ \alpha_2$.

Proof)

The unit cost functions for the final good and two intermediate goods are given by:

$$e(p_1, p_2) = \varepsilon_f\, p_1^{\ m} p_2^{\ 1-m} \,,$$

$$d_j(R, w) = \varepsilon_j R^{\alpha_j} w^{1-\alpha_j} \ (j = 1, 2),$$

where

$$\varepsilon_f = A_f^{\ -1} m^{-m} (1-m)^{-(1-m)} \,,$$

$$\varepsilon_j = A_j^{\ -1} \alpha_j^{\ -\alpha_j} (1-\alpha_j)^{-(1-\alpha_j)} \,.$$

Zero profits in the production of two intermediate goods require that $p_j = d_j(R,w)$. Plugging these equations into the numeraire constraint, $e(p_1, p_2) = 1$,

$$\varepsilon_f \varepsilon_1{}^m \varepsilon_2{}^{1-m} R^\alpha w^{1-\alpha} = 1, \tag{A1}$$

where

$$\alpha = m\alpha_1 + (1-m)\alpha_2 .$$

Equation (7) can be expressed as:

$$\frac{p_1 q_1}{p_2 q_2} = \frac{p_1 e_1}{p_2 e_2} = \frac{m}{1-m} .$$

From this, we can derive the following equations.

$$\begin{aligned} R &= \alpha v k^{-1}, \\ w &= (1-\alpha)v. \end{aligned} \tag{A2}$$

v is income per effective worker. Thus α is equal to capital share in income. Plugging (A2) into (A1), we obtain that

$$v = Ak^\alpha ,$$

where

$$A = \varepsilon_f{}^{-1} \varepsilon_1{}^{-m} \varepsilon_2{}^{-(1-m)} \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} .$$

3. *In the neoclassical growth model of section II-B where all the production functions are of Cobb-Douglas form,* $r - n - g - \phi < g_k$ *if and only if* $\dfrac{R^*}{\theta} > \alpha(\delta + n + g)$.

Proof)

If we plug the consumption function into equation (9) and divide both sides by k ,

$$g_k - (r - n - g - \phi) = \frac{1}{k} w(1 - \phi \frac{\eta}{w}) \, ,$$

where

$$\phi \frac{\eta}{w} = \frac{\int_t^\infty \exp[-\int_t^s (r - n - g - g_w) dv] \; ds}{\int_t^\infty \exp[-\int_t^s (r - n - g - g_c) dv] \; ds} \, . \tag{A3}$$

g_w is the growth rate of wage. Since $w = (1-\alpha)\, A\, k^{\alpha}$, this is equal to the growth rate of output $g_y = \alpha\, g_k$.

We can show that on the convergent path with $k(0) < k^*$, $g_c > g_y$ iff $\frac{R^*}{\theta} > \alpha(\delta + n + g)$.[1] By (A3), $\phi \frac{\eta}{w} < 1$ iff $\frac{R^*}{\theta} > \alpha(\delta + n + g)$.

***4**. Log-linearzing the Ramsey Model*

$$\begin{bmatrix} \dot{\log c} \\ \dot{\log k} \end{bmatrix} = \begin{bmatrix} 0 & -(1-\alpha)\dfrac{R^*}{\theta} \\ -(\dfrac{R^*}{\alpha} - \delta - n - g) & R^* - \delta - n - g \end{bmatrix} \begin{bmatrix} \log c - \log c^* \\ \log k - \log k^* \end{bmatrix}$$

Along the stable arm,

$$\log c - \log c^* = (1-\alpha)\frac{R^*}{\theta}\frac{1}{\lambda_c}\exp[-\lambda_c T][\log k(0) - \log k^*] \, ,$$

$$\log k - \log k^* = \exp[-\lambda_c T][\log k(0) - \log k^*] \, .$$

Thus

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[1] See Appendix 2C in Barro and Sala-i-Martin (2004).

$$g_c = \frac{1-\alpha}{\lambda_c}\frac{R^*}{\theta} g_k \text{, and } \; g_k = \frac{1}{\alpha} g_y \text{ since } y = A\,k^{\alpha}.$$

5. The optimality conditions for the endogenous growth model

By constructing a Hamiltonian, we can show that the following should hold on the optimal path.

$$\eta \; g'(z) = \pi \, . \tag{A4}$$

$$\dot{\eta} = (r - n - g(z))\eta - w + \pi \; z \, , \tag{A5}$$

$$\dot{c} = (\frac{1}{\theta}(r - \rho) - g(z))c \, , \tag{A6}$$

$$\dot{k} = (r - n - g(z))k + w - c - \pi \; z \, . \tag{A7}$$

Note that $z = z(\pi, k)$. Differentiate equation (A4) with respect to time and eliminate the terms $\dot{\eta}$ and $\dot{k}$ using equations (A5) and (A7). Solving for $\dot{\pi}$, we obtain equation (25). To obtain equation (27), note that $r\,k + w = y + \pi z - \delta k$.

6. Indeterminacy and the q theory of investment in the endogenous growth model

The returns on physical and human capital are equalized (equation 31), and a household is indifferent between accumulating physical capital and human capital. In a closed economy, this tie on the demand side is broken by the supply side constraint that $z_i = z(\pi, k_i)$. In an open economy, the outputs of two intermediate goods do not have to be equal to their domestic demands, the function $z(\pi, k_i)$ is not defined. Introducing adjustment costs in human capital accumulation breaks indeterminacy by introducing a demand function for z_i. Let q be the shadow price of (installed) human capital divided by π. Then $z^d(q)$, the demand for z, is a strictly increasing function of q. q in each country evolves according to the following equation:

$$\dot{q}_i = (r(\pi) - \dot{\pi}/\pi - n - g(z^d(q_i))q_i - w(\pi) + \pi \ z^d(q_i)\,.$$

Because π is identical in all countries, the law of motion for q_i is identical in all countries. If every country reaches a balanced growth steady state with $\dot{q}_i = 0$, the steady state value of q_i is identical, and thus, the entire path of q_i is identical in all countries, which implies an identical level of z^d in all countries.

7. Log-linearizing the endogenous growth model

From equations (31), (26) and (27), we can obtain the following log-linearized system.

$$\begin{bmatrix} \overset{\bullet}{\log \pi} \\ \overset{\bullet}{\log c} \\ \overset{\bullet}{\log k} \end{bmatrix} = \begin{bmatrix} R'\pi^* - w' + w^*\pi^{*-1} & 0 & 0 \\ (\frac{1}{\theta}R' - z_\pi)\pi^* & 0 & -z_k k^* \\ y_\pi \pi^* k^{*-1} - z_\pi \pi^* & -c^* k^{*-1} & y_k - z_k k^* - (\delta + n + g^*) \end{bmatrix} \begin{bmatrix} \log\pi - \log\pi^* \\ \log c - \log c^* \\ \log k - \log k^* \end{bmatrix}. \qquad \text{(A8)}$$

Using the properties of a 2x2 model, we can show that

$$v = R\ k + w\,,$$

$$z = v_\pi = R'k + w'\,,$$

$$z_k = v_{\pi\, k} = v_{k\, \pi} = R'\,,$$

$$y_\pi + z_\pi\ \pi = 0\,,$$

$$v_k = y_k + z_k\ \pi = R.$$

In steady state,

$$c^* = (R^* - \delta - n - g^*)(k^* + \pi^*) . \tag{A9}$$

$$\frac{w^*}{\pi^*} = R^* - \delta + \delta_H - n . \tag{A10}$$

$$z^* = \delta_H + g^* . \tag{A11}$$

Using these equations, we rewrite the terms in the matrix as follows.

$$\begin{aligned}
R'\pi^* - w' + w^*\pi^{*-1} &= R'\pi^* - (z^* - R'k^*) + R^* - \delta + \delta_H - n \\
&= R'(k^* + \pi^*) + R^* - \delta - n - g^* . \\
y_\pi - z_\pi k^* &= -z_\pi(\pi^* + k^*). \\
y_\pi \pi^* k^{*-1} - z_\pi \pi^* &= -z_\pi \pi^* \pi^* k^{*-1} - z_\pi \pi^* \\
&= -z_\pi \pi^*(k^* + \pi^*)k^{*-1} . \\
y_k - z_k k^* - (\delta + n + g^*) &= (R^* - z_k \pi^*) - z_k k^* - (\delta + n + g^*) \\
&= -R'(\pi^* + k^*) + R^* - \delta - n - g^* .
\end{aligned}$$

Plugging these expressions into system (A8), we obtain

$$\begin{bmatrix} \dot{\log\pi} \\ \dot{\log c} \\ \dot{\log k} \end{bmatrix} = \begin{bmatrix} R'(k^*+\pi^*)+R^*-\delta-n-g^* & 0 & 0 \\ (\frac{1}{\theta}R'-z_\pi)\pi^* & 0 & -R'k^* \\ -z_\pi\pi^*(k^*+\pi^*)k^{*-1} & -(k^*+\pi^*)(R^*-\delta-n-g)k^{*-1} & -R'(\pi^*+k^*)+R^*-\delta-n-g^* \end{bmatrix} \begin{bmatrix} \log\pi - \log\pi^* \\ \log c - \log c^* \\ \log k - \log k^* \end{bmatrix} . \tag{A12}$$

From the structure of the matrix, we can easily derive the three eigenvalues, μ_1, μ_2 and μ_3.

8. *In the endogenous growth model where the final good, two intermediate goods and education are produced by Cobb-Douglas technologies,*

$$\lambda_c = \frac{1-(\alpha-\beta)}{\alpha-\beta} R + \frac{\beta}{\alpha-\beta}(\delta_H - \delta - n) .$$

α is capital share in the production of the final good and β is capital share in education.

Proof)

Under the assumption that all production functions are of Cobb-Douglas form, we can show that for some positive constants b_R and b_W,

$$R = b_R \pi^{-\frac{1-\alpha}{\alpha-\beta}}, \quad w = b_w \pi^{\frac{\alpha}{\alpha-\beta}}.$$

$$R' = -\frac{1-\alpha}{\alpha-\beta}\frac{R}{\pi}, \quad w' = \frac{\alpha}{\alpha-\beta}\frac{w}{\pi}. \tag{A13}$$

Thus

$$\lambda_c = -(R'(k^* + \pi^*) + R^* - n - \delta - g^*)$$

$$\begin{aligned} &= -(R'\pi^* - w' + \frac{w^*}{\pi^*}) \\ &= \frac{1-\alpha}{\alpha-\beta}R + \frac{\beta}{\alpha-\beta}\frac{w^*}{\pi^*} \\ &= \frac{1-(\alpha-\beta)}{\alpha-\beta}R + \frac{\beta}{\alpha-\beta}(\delta_H - \delta - n). \end{aligned}$$

9. *In the log-linearized endogenous growth model where $\theta = 1$,*

$$\lambda_o = \frac{(\frac{\pi^* z^*}{v^*} - \frac{R'\pi}{R})m_\pi + \frac{R^* k^*}{v^*} - m_c}{\frac{\pi^* z^*}{v^*}m_\pi + \frac{R^* k^*}{v^*}} \quad x,$$

where $x = \frac{1}{T}(\log v(T) - \log v(0))$.

Proof)

$$\lambda_o = \frac{1}{T}\int_0^t (g_v - g_R - g_c)ds]$$

$$= \frac{1}{T}[(\log v(T) - \log v(0)) - (\log R(T) - \log R(0)) + \log(c(T) - \log c(0))]. \qquad \text{(A14)}$$

Log-linearizing $v(\pi, k)$ with the facts that $v_\pi = z$ and $v_R = k$, and equation (34),

$$\log v(T) - \log v(0) \approx (\frac{\pi^* z^*}{v^*} m_\pi + \frac{R^* k^*}{v^*})(\log k(T) - \log k(0)).$$

For $R(\pi)$,

$$\log R(T) - \log R(0) \approx \frac{R'\pi^*}{R^*} m_\pi (\log k(T) - \log k(0)).$$

From equation (35),

$$\log c(T) - \log c(0) \approx m_c (\log k(T) - \log k(0)).$$

Thus

$$\log R(T) - \log R(0) \approx \frac{\frac{R'\pi^*}{R^*} m_\pi}{\frac{\pi^* z^*}{v^*} m_\pi + \frac{R^* k^*}{v^*}} (\log v(T) - \log v(0)),$$

$$\log c(T) - \log c(0) \approx \frac{m_c}{\frac{\pi^* z^*}{v^*} m_\pi + \frac{R^* k^*}{v^*}} (\log v(T) - \log v(0)).$$

Plugging these equations into equation (A14),

$$\lambda_o \approx \frac{(\frac{\pi^* z^*}{v^*} - \frac{R'\pi}{R}) m_\pi + \frac{R^* k^*}{v^*} - m_c}{\frac{\pi^* z^*}{v^*} m_\pi + \frac{R^* k^*}{v^*}} \frac{1}{T} (\log v(T) - \log v(0)).$$

***10**. In the endogenous growth model where the final good, two intermediate goods and education are produced by Cobb-Douglas technologies,*

$$\frac{R'\pi}{R} = -\frac{1-\alpha}{\alpha - \beta}, \qquad \text{(A15)}$$

$$\frac{R^*k^*}{\pi^*} = \frac{1}{1-\alpha}[\alpha(R^*-\delta-n-g^*)+\beta\ z^*], \tag{A16}$$

$$\frac{\pi^*z^*}{v^*} = \frac{(1-\alpha)z^*}{R^*-\delta-n-g^*+(1-\alpha+\beta)\ \ z^*}, \tag{A17}$$

$$\frac{R^*k^*}{v^*} = \frac{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}{R^*-\delta-n-g^*+(1-\alpha+\beta)\ \ z^*}, \tag{A18}$$

$$m_\pi = \frac{(\alpha-\beta)[R^*-\alpha(\delta+n+g^*)+\beta\ z^*+(1-\alpha)R^*+\beta(R^*-\delta-n-g^*+z^*)]}{\dfrac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}[R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\dfrac{1}{\theta}(1-\alpha)(\alpha-\beta)^2R^*\dfrac{R^*-\delta-n-g^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}}, \tag{A19}$$

$$m_c = \frac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*+\dfrac{2}{\theta}(1-\alpha)(\alpha-\beta)R^*}{\dfrac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}[R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\dfrac{1}{\theta}(1-\alpha)(\alpha-\beta)^2R^*\dfrac{R^*-\delta-n-g^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}}. \tag{A20}$$

Proof)

Equation (A15) follows from equation (A13). Note that

$$z = R'k + w' = \frac{R'\pi}{R}\frac{Rk}{\pi} + w' = -\frac{1-\alpha}{\alpha-\beta}\frac{Rk}{\pi} + \frac{\alpha}{\alpha-\beta}\frac{w}{\pi}\ . \tag{A21}$$

In steady state,

$$\frac{w^*}{\pi^*} = R^*-\delta-n-g^*+z^*\ .$$

Plugging the real wage in equation (A21), we obtain equation (A16). Then equation (A17) follows from noting that $\dfrac{\pi^*z^*}{v^*} = \dfrac{z^*}{\dfrac{R^*k^*}{\pi^*}+\dfrac{w^*}{\pi^*}}$. Similarly, equation (A18) follows from the equation

$$\frac{R^*k^*}{v^*} = \frac{\dfrac{R^*k^*}{\pi^*}}{\dfrac{R^*k^*}{\pi^*}+\dfrac{w^*}{\pi^*}}\ .$$

Now we prove the following equations.

$$z_{\pi}(k^*+\pi^*) = \frac{1}{(1-\alpha)(\alpha-\beta)^2}\frac{1}{R^*}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][R^*-\alpha(\delta+n+g^*)+\beta\ z^*],$$

$$R'(k^*+\pi^*) = -\frac{1}{\alpha-\beta}[R^*-\alpha(\delta+n+g^*)+\beta\ z^*]\ .$$

Proof)

$$z_{\pi}\pi^* = v_{\pi\pi}\pi^* = (R''k + w'')\pi^*$$
$$= \frac{1-\alpha}{\alpha-\beta}\frac{1-\beta}{\alpha-\beta}\frac{R^*k^*}{\pi^*} + \frac{\alpha}{\alpha-\beta}\frac{\beta}{\alpha-\beta}\omega$$
$$= \frac{1}{\alpha-\beta}\frac{1-\beta}{\alpha-\beta}[\alpha(R^*-\delta-n-g^*)+\beta\ z^*] + \frac{\alpha}{\alpha-\beta}\frac{\beta}{\alpha-\beta}[R^*-\delta-n-g^*+z^*]$$
$$= \frac{1}{(\alpha-\beta)^2}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ \ z^*].$$

$$z_{\pi}(k^*+\pi^*) = z_{\pi}\pi^*(\frac{k^*}{\pi^*}+1)$$
$$= \frac{1}{(\alpha-\beta)^2}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][\frac{1}{1-\alpha}\frac{1}{R^*}(\alpha(R^*-\delta-n-g^*)+\beta\ z^*)+1]$$
$$= \frac{1}{(\alpha-\beta)^2}\frac{1}{1-\alpha}\frac{1}{R^*}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][(\alpha(R^*-\delta-n-g^*)+\beta\ z^*)+(1-\alpha)R^*]$$
$$= \frac{1}{(1-\alpha)(\alpha-\beta)^2}\frac{1}{R^*}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][R^*-\alpha(\delta+n+g^*)+\beta\ z^*].$$

$$R'(k^*+\pi^*) = R^*\frac{R'\pi^*}{R^*}(\frac{k^*}{\pi^*}+1)$$
$$= -R^*\frac{1-\alpha}{\alpha-\beta}(\frac{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}{(1-\alpha)R^*}+1)$$
$$= -\frac{1}{\alpha-\beta}(\alpha(R^*-\delta-n-g^*)+\beta\ z^*+(1-\alpha)R^*)$$
$$= -\frac{1}{\alpha-\beta}(R^*-\alpha(\delta+n+g^*)+\beta\ z^*).$$

Using the results above, we can finally calculate m_{π} and m_c in the following way. The Uzawa-Lucas case can be obtained by putting $\beta = 0$ and $\theta = 1$.

$$m_{\pi}=\frac{-\frac{k^*}{\pi^*}[2R'(k^*+\pi^*)+R^*-n-\delta-g^*]}{z_{\pi}(k^*+\pi^*)+\frac{1}{\theta}(R^*-n-\delta-g^*)}$$

$$=\frac{\frac{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}{(1-\alpha)R^*}[\frac{2}{\alpha-\beta}(R^*-\alpha(\delta+n+g^*)+\beta\ z^*)-(R^*-\delta-n-g^*)]}{\frac{1}{(1-\alpha)(\alpha-\beta)^2}\frac{1}{R^*}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\frac{1}{\theta}(R^*-n-\delta-g^*)}$$

$$=\frac{\frac{2}{\alpha-\beta}(R^*-\alpha(\delta+n+g^*)+\beta\ z^*)-(R^*-\delta-n-g^*)}{\frac{1}{(\alpha-\beta)^2}\frac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}[R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\frac{1}{\theta}(1-\alpha)R^*\frac{R^*-n-\delta-g^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}}$$

$$=\frac{(\alpha-\beta)[2(R^*-\alpha(\delta+n+g^*)+\beta\ z^*)-(\alpha-\beta)(R^*-\delta-n-g^*)]}{\frac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}[R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\frac{1}{\theta}(1-\alpha)(\alpha-\beta)^2R^*\frac{R^*-\delta-n-g^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}}$$

$$=\frac{(\alpha-\beta)[R^*-\alpha(\delta+n+g^*)+\beta\ z^*+(1-\alpha)R^*+\beta(R^*-\delta-n-g^*+z^*)]}{\frac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}[R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\frac{1}{\theta}(1-\alpha)(\alpha-\beta)^2R^*\frac{R^*-\delta-n-g^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}}.$$

$$m_c=\frac{k^*(z_{\pi}-\frac{2}{\theta}R')}{z_{\pi}(k^*+\pi^*)+\frac{1}{\theta}(R^*-\delta-n-g^*)}$$

$$=\frac{\frac{k^*}{\pi^*}z_{\pi}\pi^*-\frac{2}{\theta}\frac{R^*k^*}{\pi^*}\frac{R'\pi^*}{R^*}}{z_{\pi}(k^*+\pi^*)+\frac{1}{\theta}(R^*-\delta-n-g^*)}$$

$$=\frac{\frac{1}{(1-\alpha)(\alpha-\beta)^2}\frac{1}{R^*}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][\alpha(R^*-\delta-n-g^*)+\beta\ z^*]+\frac{2}{\theta}\frac{1}{\alpha-\beta}[\alpha(R^*-\delta-n-g^*)+\beta\ z^*]}{\frac{1}{(1-\alpha)(\alpha-\beta)^2}\frac{1}{R^*}[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\frac{1}{\theta}(R^*-\delta-n-g^*)}$$

$$=\frac{[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][\alpha(R^*-\delta-n-g^*)+\beta\ z^*]+\frac{2}{\theta}(1-\alpha)(\alpha-\beta)R^*[\alpha(R^*-\delta-n-g^*)+\beta\ z^*]}{[\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*][R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\frac{1}{\theta}(1-\alpha)(\alpha-\beta)^2R^*(R^*-\delta-n-g^*)}$$

$$=\frac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*+\frac{2}{\theta}(1-\alpha)(\alpha-\beta)R^*}{\frac{\alpha(R^*-\delta-n-g^*)+(1+\alpha-\beta)\beta\ z^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}[R^*-\alpha(\delta+n+g^*)+\beta\ z^*]+\frac{1}{\theta}(1-\alpha)(\alpha-\beta)^2R^*\frac{R^*-\delta-n-g^*}{\alpha(R^*-\delta-n-g^*)+\beta\ z^*}}.$$