Population Growth and Local Home Environment Externality in an Endogenous Growth Model with Two Engines of Growth*

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Abstract

This paper presents an endogenous growth model with Romer-type R&D process and two types of human capital accumulation process. We will introduce an inter-generational spillover of human capital due to "local home environment externality" conceptualized by Galor and Tsiddon (1997a; 1997b) so that human capital can be accumulated by population growth in addition to by human capital investment. Depending on the intensity of the externality, the model will generate a negative relationship between the population growth rate and the per capita GDP growth rate, which is also present in the data. We also obtain a paradoxical result that home education, which potentially accelerates the accumulation of human capital without any cost, could be harmful against the economic growth rate.

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1 Introduction

Under the presumption that population grows in the real world, this paper presents an endogenous growth model in which we consider Romer (1990) type R&D process and two types of human capital accumulation process. Our chief concern in this paper is to clarify how population growth affects economic growth, highlighting the role of home education which is available without any cost before economic agents enter the production process. We will apply "local home environment externality" conceptualized by Galor and Tsiddon (1997a; 1997b) into our continuous time model. We will find that the population growth rate can be negatively correlated with the per capita GDP growth rate, which is also present in the data. Interestingly we also find that home education, which potentially accelerates the accumulation of human capital, could be harmful against the economic growth rate. This result will be a caveat to policy makers that if a society relies on home education too much as a source of human capital accumulation, then the economy may be trapped in a slowly growing path.

The intensity of the local home environment externality which is given exogenously in the paper is considered to reflect the society's attitude toward home education.[1] If it is high, then the economy has cultural or institutional backgrounds in favor of home education. Depending on the intensity of the externality, the model will generate a negative relationship between the population growth rate and the per capita GDP growth rate.[2] Hence, our model can explain the real world phenomena as Dalgaard and Kreiner (2001) and Strulik (2005) and exhibits contrast with previous R&D based endogenous growth models such as Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), Howitt (1999), Segerstrom (2000) and Jones (1995a; 1995b) in which the per capita GDP growth rate is positively correlated with the population growth rate.

The presence of the local home environment externality provides another interesting outcome to our model that employs the Jones technology in the R&D process. Usually, the optimal solution with respect to human capital investment is "interior" in front of the Jones technology. However, in this model a "corner" solution is possible because aggregate human capital augments even

[1] Also, we may be able to interpret the externality from the view point of Azariadis and Drazen (1990) as the "inter-generational spillover of human capital".

[2] Empirical studies such as Kelley (1988), Kelley and Schmidt (1995) and Ahituv (2001) report a negative (or no) correlation between the population growth rate and the per capita GDP growth rate. See also Galor and Weil (2000) and Galor (2005) that document in more depth the relationship between the population growth rate and the technological development.

without human capital investment. Under plausible parameter sets, we find with a numerical method that both of corner and interior solutions are feasible along the steady growth paths. As far as we know, this is the first paper that derives multiple equilibria in a model with two engines of growth and with the Jones technology.[3] We will argue that our model will have potential to give a theoretical reason to the polarization phenomenon discussed by Krugman (1991), Lucas (1993) and Howitt (1994), among others.

Finally, with our model that has multiple equilibrium paths, we obtain a paradoxical result that the economy with stronger externality may be trapped in a slowly growing path. This is because when the local home environment externality overcomes certain threshold level, the internal solution path loses its local stability. Hence, the internal solution path cannot be supported as the optimal path and the economy is always trapped into a slowly growing path (corner solution path). This finding will be a dilemma for policy makers: potentially, they can stimulate the economic growth rate by manipulating policies to improve social backgrounds for home education, but the model holds that it is not always the case.

The paper is organized as follows. In Section 2, we set up the model. Steady growth paths are analyzed in Section 3. Stability analyses are conducted in Section 4. Section 5 concludes.

2 The Model

Our model adopts a Romer-type (1990) R&D production structure, Benhabib-Perli-Xie-type (1994) intermediate goods and final goods production structures and Uzawa (1965) and Lucas (1988) type human capital accumulation process. There are three production sectors: final goods sector, intermediate goods sector, and R&D sector; and four factors: raw labor, human capital, physical capital, and knowledge measured by the variety of intermediate goods.

We consider a continuous time model and we will omit the time script throughout the paper if there is no fear of confusion.

[3] See, for comparisons, Arnold (1998), Funke and Strulik (2000) and Strulik (2005).

2.1 Final Goods Production Sector

The final goods sector is competitive. The production function is homogenous of degree one and is given by

$$Y = L^{\alpha} H_Y^{\beta} \left(\int_0^A x(j)^{\frac{\gamma}{\zeta}} dj \right)^{\zeta}, \qquad \alpha, \beta, \gamma \in (0,1), \ \zeta \geq 1, \text{ and } \quad \alpha + \beta + \gamma = 1,$$

where Y, H_Y, L denote the amount of final goods production, human capital employed in the final goods sector, and the raw labor force, respectively. $x(j)$ denotes the amount of intermediate goods supplied by an intermediate goods firm with index j while the variety of the cluster in the intermediate goods sector is measured by A.

The first order conditions (FOCs) in this sector are given as $\frac{\partial Y}{\partial L} = w_L$, $\frac{\partial Y}{\partial H_Y} = w_Y$, and $\frac{\partial Y}{\partial x(j)} = p(j)$, where w_L is the competitive wage paid to the raw labor force, w_Y is the wage paid to human capital devoted to the final goods sector, and $p(j)$ is the price of intermediate good supplied by an intermediate good firm with index j.

2.2 Intermediate Goods Production Sector

Intermediate goods are used to produce the final goods. They are assumed to be supplied monopolistically. One unit of the intermediate good is produced by η units of physical capital, into which the final goods can be translated by one-to-one manner. Hence, the profit of an intermediate firm with index j is given by $\pi(j) = p(j)x(j) - r\eta x(i)$, where r denotes the rental price of capital. By solving the optimization problem of the intermediate goods firm with the optimal conditions in the final goods sector, the following conditions result:

$$x(j) = \left[\frac{\gamma^2}{r\eta\zeta} \right]^{\frac{1}{1-\frac{\gamma}{\zeta}}} \left[L^{\alpha} H_Y^{\beta} \left(\int_0^A x(j)^{\frac{\gamma}{\zeta}} dj \right)^{\zeta - 1} \right]^{\frac{1}{1-\frac{\gamma}{\zeta}}}, \quad \text{and} \quad p(j) = \frac{r\eta\zeta}{\gamma}.$$

In this paper we assume the symmetry in the intermediate goods sector. This assumption results in the equality of the size of intermediate goods firms. Hence, we have $K = \int_0^A \eta x(j) dj = \eta A x$, where K denotes the supply of physical capital. From this condition and the structure of the final goods production sector, we obtain the optimal production of the final goods and the market prices as

$$Y = \eta^{-\gamma} L^{\alpha} H_Y^{\beta} A^{\zeta - \gamma} K^{\gamma}, \tag{1}$$

and

$$r = \frac{\gamma^2}{\zeta} \frac{Y}{K}, \quad w_L = \alpha \frac{Y}{L}, \quad w_Y = \beta \frac{Y}{H_Y} \quad \text{and} \quad \pi = \frac{\gamma(\zeta - \gamma)}{\zeta} \frac{Y}{A}. \tag{2}$$

2.3 R&D Activities

Following the literature, the R&D process is set up as a variety-creating process in the intermediate goods sector. We introduce the Jones technology in the process so that the evolution of a new variety ($\dot{A}$) is given by

$$\dot{A} = BA^{\chi} H_A^{\phi}, \qquad B > 0, \quad \phi \in (0,1), \quad \chi \in [0,1), \tag{3}$$

where H_A denotes the amount of human capital supplied to the R&D activities. The creation of a new variety exhibits the Inada property: $\lim_{H_A \to 0} \frac{\partial \dot{A}}{\partial H_A} = \infty$ and $\lim_{H_A \to \infty} \frac{\partial \dot{A}}{\partial H_A} = 0$. By this structure, the model has no scale effects and we can introduce the population growth into the model.

The term of patent for a newly created variety is assumed to be permanent. Hence, the value of R&D (denoted as v) can be designated as the present value of perpetual monopoly profits: $v(t) \equiv \int_t^{\infty} e^{-\int_t^{\tau} r(s)ds} \pi(\tau) d\tau$. By differentiating this equation with respect to time (t) and by applying the Leibniz's rule, we obtain the well-known no-arbitrage condition which is given as

$$rv = \pi + \dot{v}. \tag{4}$$

Finally, free entry into the R&D sector is secured so that the cost and the benefit of the R&D activity be equal, which provides the following condition

$$v\dot{A} = w_A H_A,$$

where w_A is the wage paid to human capital devoted to the R&D sector. In equilibrium, $w_A = w_Y = w_H$ must hold. Here, w_H can be interpreted as the market price of human capital. Substituting the above equation into (3) gives

$$v = w_H \frac{H_A^{1-\phi}}{BA^{\chi}}. \tag{5}$$

2.4 Population Growth and Human Capital Accumulation

In our model, representative agent is endowed with one unit of raw labor force and a certain amount of human capital. As stated above, raw labor force is in-elastically supplied to the final goods production sector. Furthermore, the population of the economy grows at an exogenously given rate n. The population growth enhances the supply of row labor force (L) constantly.

In this economy, the evolution of *aggregate* human capital depends on two factors. The first element is human capital investment through Uzawa-Lucas technology. And the second element is given by the assumptions of population growth and of spillover effects in human capital accumulation process. Assume that we do not consider the second element. Then, the evolution of aggregate human capital will be obtained as

$$\dot{H} = bH_H, \quad b > 0$$

where H and H_H are the amount of human capital and human capital supplied to the Uzawa-Lucas human capital creation process, respectively. b measures the efficiency in Uzawa-Lucas technology.

In this paper, we introduce spillover effects of human capital among generations. This effect is conceptualized by Galor and Tsiddon (1997a; 1997b) as the "local home environment externality" in their discrete time model. Hence, newly born agents are endowed with positive amount of human capital since they receive home education without any cost before they enter into the production process. Although the intensity of parents' efforts to transfer human capital to children, that is, the intensity of home education, is endogenously determined in Galor and Tsiddon's models, we will regard it exogenously given. The intensity of the externality in this paper will reflect society's attitude toward home education. If it is high, then the economy has cultural or institutional backgrounds in favor of home education. We also suggest that it can be a parameter which education authority will regard as a policy target.

By taking the local home environment externality into consideration, the evolution of aggregate human capital can be written as

$$\dot{H} = bH_H + (1-\delta)nH, \quad \delta \in [0,1] \tag{6}$$

where δ represents the intensity of local home environment externality. If δ has a smaller value, then the economy has stronger cultural or institutional backgrounds in favor of home education. H is the current amount of human capital in the economy and nH captures the instantaneous amount of human capital augmented because of population growth when new agents have as much human capital as old agents. See here that when $\delta \in [0,1)$ (spillover case), aggregate human capital increases even without human capital investment ($H_H = 0$). On the other hand, when $\delta = 1$ (no spillover case), (6) reduces to $\dot{H} = bH_H$ and one needs human capital investment ($H_H > 0$) to increase aggregate human capital.

2.5 Optimization Problem of Agents

The utility of a representative agent is given by

$$\int_0^\infty \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \quad (\sigma > 0),$$

where c, ρ and σ denote the consumption by a representative agent in the dynasty,[4] the subjective discount rate, and the inverse of the elasticity of intertemporal substitution, respectively. Throughout the paper we will impose the following condition about b and ρ to ensure positive investments on human capital if there are no spillover effects in human capital accumulation process

condition 1: $b - \rho > 0$.

The budget constraint of financial assets in per capita terms is given by

$$\dot{k} = rk + w_H(h_A + h_Y) + w_L - c - nk, \tag{7}$$

where k $(= K/L)$ is the per capita financial assets. h_A $(= H_A/L)$ and h_Y $(= H_Y/L)$ represent the per capita human capital devoted to the R&D activities and the per capita human capital devoted to produce the final goods, respectively.

By considering the population growth and the exogenous local home environment externality, we can obtain the per capita human capital evolution when we divide through (6) by L as

$$\dot{h} = bh_H - \delta nh, \tag{8}$$

where h $(= H/L)$ and h_H $(= H_H/L)$ are the per capita human capital and the per capita human capital supplied to the human capital creation process, respectively.[5]

From the objective function and two constraints, the optimal condition about consumption is the usual Keynes-Ramsey rule given by

$$\sigma\frac{\dot{c}}{c} = r - n - \rho. \tag{9}$$

In addition, the optimal condition for the human capital wage reads

$$\frac{\dot{w_H}}{w_H} \leq r - b - (1-\delta)n, \quad \text{with equality whenever} \quad h_H > 0. \tag{10}$$

[4] Hence-force, per capita variables are denoted by lower-case letters.

[5] We can interpret (8) as the following way. If $\delta = 0$ (the spillover is perfect), (8) reduces to $\dot{h} = bh_H$, which means that the per capita human capital is not diluted even with the presence of growing population. This is a simple reflection of the fact that when $\delta = 0$, the newly born agents have the same amount of human capital as their ancestor have. In reality, the spillover effect will not be perfect so that in per capita term the human capital decreases due to the population growth. This effect is captured by $-\delta nh$ in (8). Notice that in the evolution of financial wealth the diluting effect of population growth is perfect so that in (7) we have the term of $-nk$ instead of $-\delta nh$.

This condition indicates that the growth rate in human capital wage must be sufficiently high compared to the interest rate in order to ensure positive investment in the human capital creation process.

Finally, the transversality conditions are given as follows:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t}\mu_t h_t = 0, \tag{11}$$

where λ and μ are the shadow prices of per capita financial assets (k) and per capita human capital (h), respectively.

The equilibrium dynamics of the economy can be depicted by (2), (4), (5), (7) – (11). Finally, the market clearing condition of human capital imposes $H = H_Y + H_A + H_H$. For later reference, we define $u_A \equiv H_A/H$, $u_H \equiv H_H/H$ and $u_Y \equiv H_Y/H$ so that $u_A + u_Y + u_H = 1$.

3 Analyses of the Steady Growth Paths

In this section we confine our attention to the case of the steady growth path (SGP), and derive some features of the model. In the present model, we have two SGPs; one path for the internal solution case (with $H_Y > 0$, $H_A > 0$, $H_H > 0$) and the other path for the corner solution case (with $H_Y > 0$, $H_A > 0$, $H_H = 0$).

To understand the importance to consider the local home environment externality and population growth at the same time, with the presence of the Jones technology, assume that there is no local home environment externality. In this case, if $H_H = 0$, there is no growth of (aggregate) human capital. This situation implies that the variety in intermediate goods should be constant over time ($\dot{A}/A = 0$) along this SGP.[6] From the structure of the Jones technology and $A > 0$, this means $H_A = 0$. However, $H_A = 0$ in front of the Jones technology is not optimal because the marginal benefit of infinitesimal increase of H_A from zero is infinite with the Jones technology ($\lim_{H_A\to 0} \frac{\partial \dot{A}}{\partial H_A} = \infty$). Then, when we impose the Jones technology the internal solution case will result as the only optimal path without the local home environment externality.

In our model, local home environment externality is taken into consideration. Hence, even if $H_H = 0$ there can be positive growth in aggregate human capital. Hence, even with the Jones technology, corner solution case could be optimal under certain parameter restrictions. Moreover, it is easy to see that both SGPs can be realized under the same parameter set: we will have multiple SGPs in our model.

[6] To see this, refer to (12) below.

In the following sections, we denote the growth rate of the variable x by g_x in the internal solution case and by $\bar{g_x}$ in the corner solution case.

3.1 The Internal Solution Case

First assume the internal solution case. In this case, we have $H_H > 0$ so that (10) is satisfied with equality.

From (8), we can see that $g_H = g_{H_H}$ on the SGP. Therefore, $g_{H_A} = g_{H_Y} = g_{H_H} = g_H$ are easily derived.[7] From (3) and $g_{H_A} = g_H$, the following condition is necessary:

$$g_A = \frac{\phi}{1-\chi} g_H. \tag{12}$$

(12) relates the growth rate of variety (g_A) to the growth rate of aggregate human capital. Notice that the Jones technology affects g_A not through the *level* parameter (B) but through the efficiency parameters (χ and ϕ).

From the resource constraint of the final goods, $\dot{K} = Y - C$, we have $g_C = g_K = g_Y$, where $C \equiv cL$ is the total consumption. From this and the production function of the final goods, g_Y is obtained as

$$g_Y = \frac{1}{1-\gamma}\left\{\alpha n + \beta g_H + (\zeta - \gamma) g_A\right\}. \tag{13}$$

By combining (12) and (13), we obtain the following relationship between g_Y and g_H

$$g_Y = \frac{1}{1-\gamma}\{\alpha n + \Psi g_H\}, \tag{14}$$

where $\Psi \equiv \beta + \frac{\zeta-\gamma}{1-\chi}\phi (> 0)$.

Next, with (10) and $w_H = \beta \frac{Y}{H_Y}$, we have

$$g_Y - g_H = r - b - (1-\delta)n. \tag{15}$$

Moreover, on the SGP, (9) is written into

$$g_Y - n = \frac{1}{\sigma}(r - n - \rho). \tag{16}$$

From (15) and (16), we obtain another relationship between g_Y and g_H as

$$g_H = (1-\sigma)(g_Y - n) + (1-\delta)n + b - \rho. \tag{17}$$

As it can be seen from (14) and (17), the GDP growth rate (g_Y) and the aggregate human capital growth rate (g_H) are inter-dependently determined,

[7] The proof of uniqueness of allocation of H among H_A, H_Y and H_H on the SGP is straightforward and is available from the authors upon request.

which is not the case in Arnold (1998). This is because we consider a general CRRA felicity function rather than a log-separable one.

From (14) and (17), g_Y is obtained as

$$g_Y = \frac{\alpha + (\sigma - \delta)\Psi}{\Upsilon} n + \frac{\Psi}{\Upsilon}(b - \rho), \tag{18}$$

where $\Upsilon \equiv \alpha + \beta - (1 - \sigma)\Psi$. We can re-write (18) into a per capita form as

$$g_y = \frac{(\underline{\delta} - \delta)\Psi}{\Upsilon} n + \frac{\Psi}{\Upsilon}(b - \rho), \tag{19}$$

where g_y is the per capita GDP growth rate and $\underline{\delta} = \frac{\Psi - \beta}{\Psi} \in (0, 1)$. We can see from (19) that g_y is a linear combination of the factor of the population growth rate n and the factor of contribution of Uzawa-Lucas type human capital accumulation $b - \rho$. Hence, the signs of Υ and $(\underline{\delta} - \delta)$ determine the relation ship between n and g_y. Notice here that if $\Upsilon < 0$, then an unfamiliar implication results: in that case, $(b - \rho)$ affects g_y negatively. In this paper we will exclude this case by imposing the following condition

condition 2: $\Upsilon > 0$

With the condition 2, we can have a negative correlation between the population growth rate and the per capita GDP growth rate when $\underline{\delta} - \delta < 0$ is satisfied. Thus, in order to accommodate with empirical findings, we should impose relatively large value for δ. That is, the home education effect should not be so strong.[8] The relationship is supported by empirical findings such as Kelley (1988), Kelley and Schmidt (1995) and Ahituv (2001).

Next we should investigate conditions to ensure the internal solution obtained above. First, from the internal solution assumption ($H_H > 0$) and from (8), $\dot{h}/h = bh_H/h - \delta n > -\delta n$. With a little algebra, this condition can be re-written into

$$b > \rho - \frac{1 - \sigma}{\alpha + \beta} (\underline{\delta} - \delta) \Psi n. \tag{20}$$

Secondly, with the transversality conditions (11), a little algebra gives the other constraint as

$$b \left\{ \begin{array}{c} > \\ < \end{array} \right\} \frac{\alpha + \beta}{1 - \sigma} \Psi \rho - (\underline{\delta} - \delta) n, \quad \text{for} \left\{ \begin{array}{c} \sigma < 1 \\ \sigma > 1 \end{array} \right. . \tag{21}$$

When $\sigma = 1$, the transversality conditions are automatically satisfied. We obtain that the SGP of the internal solution can generate under the parameter space satisfying both (20) and (21).

[8] Indeed, by definition, the externality should not be so strong.

3.2 The Corner Solution Case

This section investigates the corner solution case ($H_H = 0$). By remembering that (10) is not satisfied with equality and by replicating the procedure in the previous section with the condition $H_H = 0$, we can obtain the GDP growth rate ($\bar{g}_Y$) and the per capita GDP growth rate ($\bar{g}_y$) in the corner solution case as

$$\bar{g}_Y = \frac{\Psi}{\alpha + \beta}(\underline{\delta} - \delta)n + n, \tag{22}$$

and

$$\bar{g}_y = \frac{\Psi}{\alpha + \beta}(\underline{\delta} - \delta)n. \tag{23}$$

(23) shows that the per capita GDP growth rate exhibits a *semi-endogenous growth* property: the growth rate is pinned down to the population growth rate as in the second stage models.

In turn, we should investigate parametric conditions to generate the corner solution case. This is done by examining (10) when it holds with inequality. Because $\bar{g}_{w_H} = \bar{g}_Y - \bar{g}_H$ on the SGP, the condition given by (10) can be written into

$$\bar{g}_{w_H} = \bar{g}_Y - \bar{g}_H < r - b - (1 - \delta)n. \tag{24}$$

With (9) and the condition that $\bar{g}_Y = \bar{g}_C$ on the SGP, the following condition is obtained

$$\sigma(\bar{g}_Y - n) = r - \rho - n. \tag{25}$$

In the corner solution case, $\bar{g}_H = (1 - \delta)n$. Also we can eliminate r and $\bar{g}_Y$ from (25) with (24) and with (22). Hence, we obtain the parametric condition for the corner solution case as

$$b > \rho - \frac{1 - \sigma}{\alpha + \beta}(\underline{\delta} - \delta)\Psi n. \tag{26}$$

Notice that the partition given by (26) is equivalent to the one given by (20) in the internal solution case.

Finally, a negative correlation between n and $\bar{g}_y$ can be obtained when $(\underline{\delta} - \delta) < 0$ with the condition 2, as in the internal solution case.

4 Stability of the SGPs

In this section, we will investigate the local stability of two SGPs with a numerical method. We calibrate the model in accordance with strains of previous literature and examine the stability of the dynamical system on the SGPs.

4.1 The internal solution case

In the internal solution case, the dynamical system is given by the following seven equations of seven variables consisting of $\{L, K, H, A, u_Y, u_A, v\}$

$$\dot{L} = nL, \tag{27}$$

$$\dot{K} = Y(L, H, K, A, u_Y) - C, \tag{28}$$

$$\sigma\left(\frac{\dot{C}}{C} - n\right) = r(L, H, K, A, u_Y) - \rho - n, \tag{29}$$

$$\frac{\dot{w}_H}{w_H} = r(L, H, K, A, u_Y) - b - (1-\delta)n, \tag{30}$$

$$\dot{v} + \pi = r(L, H, K, A, u_Y)v, \tag{31}$$

$$\dot{H} = b(1 - u_A - u_Y)H + (1-\delta)nH, \tag{32}$$

and

$$\dot{A} = BA^{\chi}(u_A H)^{\phi}. \tag{33}$$

Notice here that (i) u_H does not appear in the system because u_H is determined by the condition $u_A + u_Y + u_H = 1$ when u_Y and u_A are given, (ii) Y is given by $Y = \eta^{-\gamma} L^{\alpha}(u_Y H)^{\beta} A^{\zeta-\gamma} K^{\gamma}$ from (1), (iii) r is given by $r = \frac{\gamma^2}{\zeta}\frac{Y}{K}$ from (2), (iv) π is given by $\pi = \frac{\gamma(\zeta-\gamma)}{\zeta}\frac{Y}{A}$ from (2) and (v) w_H is given by $w_H = \beta\frac{Y}{u_Y H}$ from (2), or equivalently, w_H is given by the condition $v = w_H \frac{(u_A H)^{1-\phi}}{BA^{\chi}}$ from (5).

In order to describe the system with variables which are stationary on the steady growth path, we define the following variables[9]

$$\tilde{c} \equiv \frac{C}{L^{\frac{\alpha}{1-\gamma}} H^{\frac{\beta}{1-\gamma}} A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \tilde{k} \equiv \frac{K}{L^{\frac{\alpha}{1-\gamma}} H^{\frac{\beta}{1-\gamma}} A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \text{and} \quad \xi \equiv \frac{H}{A^{\frac{1-\chi}{\phi}}}. \tag{34}$$

By using these variables, the dynamical system of the internal solution case can be reduced to the system of five equations consisting of five variables: $\{\xi, \tilde{c}, \tilde{k}, u_A, u_Y\}$. By construction, these five variables are constant on the steady growth path.

The intensive form dynamical system can be obtained as follows[10]

$$\dot{\tilde{k}} = \left[\eta^{-\gamma} u_Y^{\beta} \tilde{k}^{\gamma-1} - \frac{\tilde{c}}{\tilde{k}} - \frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n + \beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)} B u_A^{\phi} \xi^{\phi}\right]\tilde{k}, \tag{35}$$

$$\begin{aligned}\dot{\tilde{c}} = {} & \left[\frac{1}{\sigma}\left\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma} u_Y^{\beta} \tilde{k}^{\gamma-1}) - \rho - n\right\} + n \right.\\ & \left. -\frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n + \beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)} B u_A^{\phi} \xi^{\phi}\right]\tilde{c},\end{aligned} \tag{36}$$

[9] See (12) for the structure of ξ, and (13) for the structure of $\tilde{c}$ and $\tilde{k}$.

[10] See appendix a for a detailed description on how we have the result.

$$\dot{\xi} = \{b(1-u_A-u_Y)+(1-\delta)n - \frac{1-\chi}{\phi}Bu_A^{\phi}\xi^{\phi}\}\xi, \tag{37}$$

$$\dot{u_A} = \left[-b(1-u_A-u_Y)+b+\phi(1-\delta)n+\left\{\chi-\frac{\gamma(\zeta-\gamma)}{\zeta\beta}\frac{u_Y}{u_A}\right\}Bu_A^{\phi}\xi^{\phi}\right]\frac{u_A}{1-\phi}, \tag{38}$$

and

$$\begin{aligned}\dot{u_Y} \quad = \quad & \Bigg\{-b(1-u_A-u_Y)+\frac{\{\alpha+\beta(1-\delta)\}n}{1-\beta} \\ & +\frac{(\zeta-\gamma)B}{1-\beta}u_A^{\phi}\xi^{\phi}+\frac{\gamma}{1-\beta}(\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}-\frac{\tilde{c}}{\tilde{k}})\Bigg\}u_Y.\end{aligned} \tag{39}$$

Next, in order to analyze the local stability of the dynamical system in the internal solution case, we will derive the steady-growth-path-values of $\{u_Y^{sgp}, u_A^{sgp}, \xi^{sgp}, \tilde{c}^{sgp}, \tilde{k}^{sgp}\}$. The SGP values are obtained as

$$u_A^{sgp} = (1-u_H^{sgp})\left\{1+\frac{(r-g_Y+g_A)\beta\zeta}{\gamma(\zeta-\gamma)g_A}\right\}^{-1}, \tag{40}$$

$$u_Y^{sgp} = 1-u_A^{sgp}-u_H^{sgp}, \tag{41}$$

$$\xi^{sgp} = \left(\frac{g_A}{B}\right)^{\frac{1}{\phi}}\frac{1}{u_A^{sgp}}, \tag{42}$$

$$\tilde{k}^{sgp} = \left(\frac{r\zeta\eta^{\gamma}}{\gamma^2(u_Y^{sgp})^{\beta}}\right)^{\frac{-1}{1-\gamma}}, \tag{43}$$

and

$$\tilde{c}^{sgp} = (\frac{r\zeta}{\gamma^2}-g_Y)\tilde{k}^{sgp}, \tag{44}$$

where r denotes the interest rate in the inner solution case and $u_H^{sgp} = \frac{g_H-(1-\delta)n}{b}$.

4.2 The Corner Solution Case

In the corner solution case, the dynamical system is given by the following six equations of six variables consisting of $\{L, K, H, A, u_Y, v\}$

$$\dot{L} = nL, \tag{45}$$

$$\dot{K} = Y(L,H,K,A,u_Y)-C, \tag{46}$$

$$\sigma\left(\frac{\dot{C}}{C}-n\right) = r(L,H,K,A,u_Y)-\rho-n, \tag{47}$$

$$\dot{v}+\pi = r(L,H,K,A,u_Y)v, \tag{48}$$

$$\dot{H} = (1-\delta)nH, \tag{49}$$

and

$$\dot{A} = BA^{\chi}\{(1-u_Y)H\}^{\phi}. \tag{50}$$

Notice here that (i) u_H and u_A do not appear in the system because $u_H = 0$ and u_A is determined by the condition $u_A + u_Y = 1$ when u_Y is given,[11] (ii) Y is given by $Y = \eta^{-\gamma} L^{\alpha}(u_Y H)^{\beta} A^{\zeta-\gamma} K^{\gamma}$ from (1), (iii) r is given by $r = \frac{\gamma^2}{\zeta}\frac{Y}{K}$ from (2), (iv) π is given by $\pi = \frac{\gamma(\zeta-\gamma)}{\zeta}\frac{Y}{A}$ from (2) and (v) v is determined by $v = w_H \frac{\{(1-u_Y)H\}^{1-\phi}}{BA^{\chi}}$ from (5) whereas w_H is given by $w_H = \beta\frac{Y}{u_Y H}$ from (2).

Likely to the internal solution case, we define the following variables in order to obtain the intensive form dynamical system in the corner solution case

$$\tilde{c}_c \equiv \frac{C}{L^{\frac{\alpha}{1-\gamma}} H^{\frac{\beta}{1-\gamma}} A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \tilde{k}_c \equiv \frac{K}{L^{\frac{\alpha}{1-\gamma}} H^{\frac{\beta}{1-\gamma}} A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \text{and} \quad \xi_c \equiv \frac{H}{A^{\frac{1-\chi}{\phi}}}, \tag{51}$$

By using these variables, the dynamical system of the corner solution case can be reduced to the system of four equations consisting of four variables: $\{\xi_c, \tilde{c}_c, \tilde{k}_c, u_{Y,c}\}$ where the subscript c denotes "corner solution case". By construction, these four variables are constant on the steady growth path.

The intensive form dynamical system in the corner solution case can be obtained as follows[12]

$$\begin{aligned} \dot{\tilde{c}}_c &= \Big[\frac{1}{\sigma}\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma} u_{Y,c}^{\beta} \tilde{k}_c^{\gamma-1}) - \rho - n\} + n \\ &\quad - \frac{\alpha}{1-\gamma} n - \frac{\beta(1-\delta)}{1-\gamma} n - \frac{(\zeta-\gamma)}{(1-\gamma)} B(1-u_{Y,c})^{\phi} \xi_c^{\phi}\Big] \tilde{c}_c, \end{aligned} \tag{52}$$

$$\dot{\tilde{k}}_c = \left[\eta^{-\gamma} u_{Y,c}^{\beta} \tilde{k}_c^{\gamma-1} - \frac{\tilde{c}_c}{\tilde{k}_c} - \frac{\alpha}{1-\gamma} n - \frac{\beta(1-\delta)}{1-\gamma} n - \frac{(\zeta-\gamma)}{(1-\gamma)} B(1-u_{Y,c})^{\phi} \xi_c^{\phi}\right] \tilde{k}_c, \tag{53}$$

$$\dot{\xi}_c = \{(1-\delta)n - \frac{1-\chi}{\phi} B(1-u_{Y,c})^{\phi} \xi_c^{\phi}\} \xi_c, \tag{54}$$

and

$$\begin{aligned} \dot{u}_{Y,c} &= \Big[\{(\phi-\beta)(1-\delta) - \alpha\} n + \frac{(\gamma-\zeta)\gamma}{\zeta} \eta^{-\gamma} u_{Y,c}^{\beta} \tilde{k}_c^{\gamma-1} + \gamma \frac{\tilde{c}_c}{\tilde{k}_c} \\ &\quad + (\chi - \zeta + \gamma - \frac{\gamma(\zeta-\gamma)}{\beta\zeta} \frac{u_{Y,c}}{1-u_{Y,c}}) B(1-u_{Y,c})^{\phi} \xi_c^{\phi}\Big] \frac{u_{Y,c}(1-u_{Y,c})}{\beta - 1 + (\phi-\beta) u_{Y,c}}. \end{aligned} \tag{55}$$

Next, in order to analyze the local stability of the dynamical system in the corner solution case, we will derive the steady-growth-path-values of $\{u_{Y,c}^{sgp}, \xi_c^{sgp}, \tilde{c}_c^{sgp}, \tilde{k}_c^{sgp}\}$. They are determined as

$$u_{Y,c}^{sgp} = \frac{\Omega}{1+\Omega}, \quad \text{where} \quad \Omega = \beta\zeta(\bar{r} - \bar{g}_Y + \bar{g}_A)/(\gamma(\zeta-\gamma)\bar{g}_A) \tag{56}$$

$$\xi_c^{sgp} = \left(\frac{\bar{g}_A}{B}\right)^{\frac{1}{\phi}} \frac{1}{(1-u_{Y,c}^{sgp})}, \tag{57}$$

[11] It is easy to see that since u_A drops off the dynamical system, the dimension of the system reduces in the corner solution case to six from seven in the internal solution case.

[12] See appendix b for a detailed derivation.

$$\tilde{k}_c^{sgp} = \left(\frac{\bar{r}\zeta\eta^{\gamma}}{\gamma^2 (u_{Y,c}^{sgp})^{\beta}} \right)^{\frac{-1}{1-\gamma}}, \tag{58}$$

and

$$\tilde{c}_c^{sgp} = (\frac{\bar{r}\zeta}{\gamma^2} - \bar{g}_Y)\tilde{k}_c^{sgp}, \tag{59}$$

where $\bar{r}$ is the interest rate in the corner solution case.

4.3 Calibration and Discussion

We calibrate our model as follows. First, following Benhabib et al. (1994), we can designate the labor share (S_L), the share of human capital (S_H) and the capital share (S_K) in the final goods sector by $S_L = \alpha/(\alpha + \beta + (\gamma^2/\zeta))$, $S_H = \beta/(\alpha+\beta+(\gamma^2/\zeta))$ and $S_K = (\gamma^2/\zeta)/(\alpha+\beta+(\gamma^2/\zeta))$, respectively. We set $S_L = 0.25$, $S_H = 0.5$ and $S_K = 0.25$ following Benhabib et al. (1994). $\zeta = 2$ is cited from them as well. These conditions require that $\alpha = 0.15$, $\beta = 0.30$, and $\gamma = 0.55$. We set the subjective discount rate ρ to be 0.04, and the population growth rate n to be 0.03. The latter figure is drawn from Kortum (1997). We set $\eta = 0.34$, which gives some 25% markup in the intermediate goods sector, as is the case of Benhabib et al. (1994). b is set to be 0.05 from Lucas (1988), Benhabib and Perli (1994) and Funke and Strulik (2000). We set $\sigma = 0.9$. ϕ and χ must be determined such that the Jones technology exhibits diminishing returns to scale and we choose $\phi = 0.3$ and $\chi = 0.5$. The level parameter in the R&D process is set to be $B = 1$.

Finally, δ should be determined. This is the parameter to determine the intensity of spillover effects in human capital accumulation process. We examine with $\delta = 0.8$ and $\delta = 0.9$. We will investigate the case $\delta = 1$ (no spillover case) in order to compare the result with those in the economy with the local home environment externality.

Table 1 shows some key economic variables which result from the parameter sets. Notice also that the condition 2 is satisfied with these parameters. The sign of $\underline{\delta} - \delta$ is negative so that the population growth affects the per capita GDP growth rate negatively, which is also present in the data. Most importantly, conditions (20), (21) and (26) are satisfied so that we have multiple SGPs under the parameter sets.

Now we examine the local stability of two SGPs.[13] In the internal solution case, the system consists of $\{\xi, \tilde{c}, \tilde{k}, u_A, u_Y\}$ and there are three jump variables

[13] The matlab codes to calculate the roots of the system are available from the authors upon requests.

and two state variables. On the other hand, in the corner solution case, the system consists of $\{\xi_c, \tilde{c}_c, \tilde{k}_c, u_{Y,c}\}$ and there are two jump variables and two state variables. Because we adopt the production structure of Benhabib et al. (1994), it seems that local indeterminacy will result.[14] However, our investigations will show that the SGP of the corner solution is robustly saddle stable, and that the SGP of the internal solution case will be either saddle stable or unstable in the sense that there are too many explosive roots. At this point, we suspect that this contrast between Benhabib et al. (1994) and the present analysis is attributable to assumptions in R&D process; there are scale effects in Benhabib et al. (1994) while we adopt the Jones technology thereby eliminating the increasing returns effects in the R&D process, which would be a chief source of indeterminacy in Benhabib et al. (1994).

Table 2 shows the eigen values of the matrix which we obtain by linearizing the dynamical systems around the SGPs when $\delta = 0.9$ and $\delta = 0.8$. As table 2 suggests, when $\delta = 0.9$ both of the SGPs are locally saddle stable. With the results it can be said that both SGPs can be realized as optimal paths. In the case where we have multiple equilibria, the equilibrium path to be realized depends on the self-fulfilling expectations of households; and hence, our result provides a reason to explain the polarization of economies discussed by Krugman (1991), Lucas (1993) and Howitt (1994). We argue that multiplicity in human capital investment decisions could cause the phenomenon.

If $\delta = 0.8$, the situation changes (see table 2). In this case, the SGP of the internal solution case cannot be realized as an optimal path because the number of unstable roots is bigger than the number of the jump variables in the system. The SGP of the corner solution case is still locally saddle stable. In this case, though the economy potentially has two SGPs, only the corner solution path will realize. In this case, our model would have consequences on the discussion by Lucas (1993): we may need a miracle to attain a higher growth rate with positive investments on human capital.[15]

It is most interesting to compare above results with the case where we have no spillover effects ($\delta = 1$). In the case of $\delta = 1$, we do not have multiple equilibria and only the internal solution case will result. The key economic variables are in table 3. As tables 1 and 3 suggest, the GDP growth rate in

[14] In the present model, there will be no mechanism to bring indeterminacy from the Lucas-type human capital production process because in the present model there is no externality of human capital in the final goods production sector. See also Behnabib and Perli (1994).

[15] The probability that the internal solution case realizes is zero in this case in the Lebesgue sense.

the economy when $\delta = 1$ is higher than that of the corner solution case when $\delta = 0.8$. Moreover, the SGP when $\delta = 1$ is locally saddle stable (see table 4). This result casts a paradox: it may seem that if there is positive externality in human capital accumulation process, the local home environment externality, then the growth rate will be higher than the one we have when there is no local home environment externality. However, our results hold that it is not always the case. The economy may be trapped in a slowly growing path without human capital investments due to the externality. This phenomenon is a dilemma for policy makers: potentially, they could stimulate the economic growth rate by manipulating policies to improve social backgrounds for home education.

Actually, the situation that the growth rate in the society may be low when the local home environment externality matters is illustrated by Galor and Weil (2000). Quoting Schultz (1964), they argue that

> $\cdots$ when productive technology has been constant [and low] for a long period of time, farmers will have learned to use their resources efficiently. Children will acquire knowledge of how to deal with this environment directly from observing their parents, and formal schooling will have little economic value $\cdots$. [Oded Galor and David N. Weil, "Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond," p810]

This argument will be, again, a reflection of difficulty to attain "modern economic growth" through the augmentation of human capital.

5 Concluding Remarks

This paper constructs an endogenous growth model with Romer type R&D process with the Jones technology and two types of human capital accumulation process of home education and higher education. Highlighting the effect of the local home environment externality, our analyses show that the population growth rate can be negatively correlated with the per capita GDP growth rate, which is also present in the data. Interestingly, we also find that home education, which potentially accelerates the accumulation of human capital, could be harmful against the economic growth rate: if a society relies on home education too much as a source of human capital accumulation, then the economy may be trapped in a slowly growing path. Hence, this is a dilemma.

As a final remark, it is important to point out that labor supply is exogenous in the model. We believe that a promising extension to our model would be to either endogenize the fertility rate or allow agents to make labor-leisure choices. We leave these tasks for future research.

6 Appendicies

6.1 Appendix a

We can derive the intensive form dynamical system in the internal solution case as the following way. First take the natural logarithm of (34) and differentiate the equations with time. Hence, we will obtain the dynamic evolution of ξ, $\tilde{c}$ and $\tilde{k}$ as

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{C}}{C} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{60}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{61}$$

and

$$\frac{\dot{\xi}}{\xi} = \frac{\dot{H}}{H} - \frac{1-\chi}{\phi}\frac{\dot{A}}{A}. \tag{62}$$

From (32), we obtain that $\frac{\dot{H}}{H} = b(1-u_A-u_Y)+(1-\delta)n$. From (33), we see $\frac{\dot{A}}{A} = Bu_A^{\phi}\xi^{\phi}$ and from (27) we have $\dot{L}/L = n$. From (29) and (2), $\frac{\dot{C}}{C}$ is obtained as

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(\frac{\gamma^2}{\zeta}\frac{Y}{K} - \rho - n) + n. \tag{63}$$

$\frac{Y}{K}$ is obtained from (1) that

$$\frac{Y}{K} = \eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}. \tag{64}$$

From the resource constraint, we have $\frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{K}$. By this condition, we have

$$\frac{\dot{K}}{K} = \eta^{\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1} - \frac{\tilde{c}}{\tilde{k}}. \tag{65}$$

Substituting these conditions into (60) – (62) ,we will have the following three equations

$$\dot{\tilde{k}} = \left[\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1} - \frac{\tilde{c}}{\tilde{k}} - \frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n+\beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)}Bu_A^{\phi}\xi^{\phi}\right]\tilde{k}, \tag{66}$$

$$\begin{aligned}\dot{\tilde{c}} \quad = \quad & \left[\frac{1}{\sigma}\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}) - \rho - n\} + n\right. \\ & \left. - \frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n+\beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)}Bu_A^{\phi}\xi^{\phi}\right]\tilde{c}, \end{aligned} \tag{67}$$

$$\dot{\xi} = \{b(1-u_A-u_Y)+(1-\delta)n - \frac{1-\chi}{\phi}Bu_A^{\phi}\xi^{\phi}\}\xi. \tag{68}$$

Next, from (30) and (5) we have

$$\frac{\dot{v}}{v}+\chi\frac{\dot{A}}{A}+(\phi-1)(\frac{\dot{u}_A}{u_A}+\frac{\dot{H}}{H}) = \frac{\dot{w}_H}{w_H} = r-b-(1-\delta)n. \tag{69}$$

Substituting the value of π from (2) into (31), we obtain the growth rate of v as follows:

$$\frac{\dot{v}}{v} = r - \frac{\gamma(\zeta-\gamma)}{\beta\zeta}\frac{u_Y}{u_A}Bu_A^{\phi}\xi^{\phi}. \tag{70}$$

Substituting $\frac{\dot{H}}{H} = b(1-u_A-u_Y)+(1-\delta)n$, $\frac{\dot{A}}{A} = Bu_A^{\phi}\xi^{\phi}$ and (70) into (69), we obtain the dynamics of u_A as

$$\dot{u_A} = \left[-b(1-u_A-u_Y)+b+\phi(1-\delta)n+\left\{\chi-\frac{\gamma(\zeta-\gamma)}{\zeta\beta}\frac{u_Y}{u_A}\right\}Bu_A^{\phi}\xi^{\phi}\right]\frac{u_A}{1-\phi}. \tag{71}$$

From (2) we have $\frac{\dot{w}_H}{w_H} = \frac{\dot{Y}}{Y} - (\frac{\dot{H}}{H}+\frac{\dot{u}_Y}{u_Y})$. With this condition and (30), the following equation is obtained

$$r-b-(1-\delta)n = \frac{\dot{w}_H}{w_H} = \frac{\dot{Y}}{Y} - (\frac{\dot{H}}{H}+\frac{\dot{u}_Y}{u_Y}). \tag{72}$$

From the production function, $\frac{\dot{Y}}{Y} = \alpha n + \beta\left(\frac{\dot{H}}{H}+\frac{\dot{u}_Y}{u_Y}\right)+(\zeta-\gamma)\frac{\dot{A}}{A}+\gamma\frac{\dot{K}}{K}$. With these arguments, we have the dynamics of u_Y as follows

$$\begin{aligned} \dot{u_Y} \;=\; & \left\{-b(1-u_A-u_Y)+\frac{\{\alpha+\beta(1-\delta)\}n}{1-\beta}\right. \\ & \left. +\frac{(\zeta-\gamma)B}{1-\beta}u_A^{\phi}\xi^{\phi}+\frac{\gamma}{1-\beta}(\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}-\frac{\tilde{c}}{\tilde{k}})\right\}u_Y. \end{aligned} \tag{73}$$

(66),(67),(68),(71) and (73) depict the dynamical system of the model.

Next we will derive the SGP values of $\{u_Y^{sgp}, u_A^{sgp}, \xi^{sgp}, \tilde{c}^{sgp}, \tilde{k}^{sgp}\}$. From (15), the interest rate on the SGP in the internal solution case is determined as

$$r = g_Y - g_H + b + (1-\delta)n. \tag{74}$$

From (31), we can have $r = \frac{\dot{v}}{v}+\frac{\pi}{v}$. Also, from (2), we have that the growth rate of π equals $g_Y - g_A$. With these conditions and the fact that the growth rates of π and v are same, we have

$$\frac{\pi}{v} = r - g_Y + g_A. \tag{75}$$

We can solve above equation with (2), (5) and $g_A = B(u_A^{sgp}\xi^{sgp})^{\phi}$ as

$$\frac{\gamma(\zeta-\gamma)g_A}{\beta\zeta}\frac{u_Y^{sgp}}{u_A^{sgp}} = r - g_Y + g_A. \tag{76}$$

From (32), on the SGP we have

$$g_H = b(1 - u_Y^{sgp} - u_A^{sgp}) + (1-\delta)n = b u_H^{sgp} + (1-\delta)n. \tag{77}$$

Then, we have $u_H^{sgp} = \frac{g_H - (1-\delta)n}{b}$ where g_H is already obtained with (17) and (18). From the resource constraint we obtain

$$u_A^{sgp} = 1 - u_Y^{sgp} - u_H^{sgp}. \tag{78}$$

Hence, from (76) and (78) we obtain u_Y^{sgp} and u_A^{sgp} respectively as

$$u_A^{sgp} = (1 - u_H^{sgp})\left\{1 + \frac{(r - g_Y + g_A)\beta\zeta}{\gamma(\zeta-\gamma)g_A}\right\}^{-1}, \tag{79}$$

and

$$u_Y^{sgp} = 1 - u_A^{sgp} - u_H^{sgp}. \tag{80}$$

ξ^{sgp} follows immediately due to $g_A = B(u_A^{sgp}\xi^{sgp})^{\phi}$ as

$$\xi^{sgp} = \left(\frac{g_A}{B}\right)^{\frac{1}{\phi}} \frac{1}{u_A^{sgp}}. \tag{81}$$

Next, from the resource constraint ($\dot{K} = Y - C$) and (2) we have

$$\frac{C}{K} = \frac{r\zeta}{\gamma^2} - g_Y = \frac{\tilde{c}^{sgp}}{\tilde{k}^{sgp}}. \tag{82}$$

Moreover, from $\frac{Y}{K} = \frac{r\zeta}{\gamma^2}$ and (1) we have

$$\eta^{-\gamma} L^{\alpha}(u_Y H)^{\beta} A^{\zeta-\gamma} K^{\gamma-1} = \frac{r\zeta}{\gamma^2}. \tag{83}$$

From the definition of $\tilde{k}$ and with a little algebra, we have

$$\tilde{k}^{sgp} = \left(\frac{r\zeta\eta^{\gamma}}{\gamma^2 (u_Y^{sgp})^{\beta}}\right)^{\frac{-1}{1-\gamma}}. \tag{84}$$

Finally, with $\tilde{k}^{sgp}$, $\tilde{c}^{sgp}$ is obtained from (82) as

$$\tilde{c}^{sgp} = (\frac{r\zeta}{\gamma^2} - g_Y)\tilde{k}^{sgp}. \tag{85}$$

6.2 Appendix b

We can derive the intensive form dynamical system in the corner solution case as the following way. First take the natural logarithm of (51) and differentiate the equations with time. Hence, we will obtain respectively the dynamic evolution of $\tilde{c}_c$, $\tilde{k}_c$ and ξ_c as

$$\frac{\dot{\tilde{c}}_c}{\tilde{c}_c} = \frac{\dot{C}}{C} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{86}$$

$$\frac{\dot{\tilde{k}}_c}{\tilde{k}_c} = \frac{\dot{K}}{K} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{87}$$

and

$$\frac{\dot{\xi}_c}{\xi_c} = \frac{\dot{H}}{H} - \frac{1-\chi}{\phi}\frac{\dot{A}}{A}. \tag{88}$$

In the corner solution case, the growth rate of aggregate human capital is determined as $\frac{\dot{H}}{H} = (1-\delta)n$ from (49). Also, the growth rate of the variety is obtained as $\frac{\dot{A}}{A} = B(1-u_{Y,c})^{\phi}\xi_c^{\phi}$ from (50) whereas $\frac{\dot{L}}{L} = n$ from (45).

From the resource constraint, we have $\frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{K}$. By this condition, we have $\bar{g}_K = \bar{g}_Y = \bar{g}_C = \bar{g}_Y$ on the SGP. From (47) and (2), $\frac{\dot{C}}{C}$ is obtained as

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(\frac{\gamma^2}{\zeta}\frac{Y}{K} - \rho - n) + n. \tag{89}$$

Finally, $\frac{Y}{K}$ is obtained as

$$\frac{Y}{K} = \eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} \tag{90}$$

With these conditions, we have

$$\begin{aligned}\dot{\tilde{c}}_c \;\; = \;\; & \Big[\frac{1}{\sigma}\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1}) - \rho - n\} + n \\ & -\frac{\alpha}{1-\gamma}n - \frac{\beta(1-\delta)}{1-\gamma}n - \frac{(\zeta-\gamma)}{(1-\gamma)}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\Big]\tilde{c}_c, \end{aligned} \tag{91}$$

$$\dot{\tilde{k}}_c = \left[\eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} - \frac{\tilde{c}_c}{\tilde{k}_c} - \frac{\alpha}{1-\gamma}n - \frac{\beta(1-\delta)}{1-\gamma}n - \frac{(\zeta-\gamma)}{(1-\gamma)}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\right]\tilde{k}_c, \tag{92}$$

and

$$\dot{\xi}_c = \{(1-\delta)n - \frac{1-\chi}{\phi}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\}\xi_c. \tag{93}$$

With (48), (2) and (5) a little algebra leads to the evolution of $u_{Y,c}$ as

$$\begin{aligned}\dot{u}_{Y,c} \;\; = & \Big[\{(\phi-\beta)(1-\delta)-\alpha\}n + \frac{(\gamma-\zeta)\gamma}{\zeta}\eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} + \gamma\frac{\tilde{c}_c}{\tilde{k}_c} \\ & +(\chi-\zeta+\gamma-\frac{\gamma(\zeta-\gamma)}{\beta\zeta}\frac{u_{Y,c}}{1-u_{Y,c}})B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\Big]\frac{u_{Y,c}(1-u_{Y,c})}{\beta-1+(\phi-\beta)u_{Y,c}}. \end{aligned} \tag{94}$$

(91) – (94) regulate the dynamics of the system.

Next we will derive the SGP values of $\{u_{Y,c}^{sgp}, \xi_c^{sgp}, \tilde{c}_c^{sgp}, \tilde{k}_c^{sgp}\}$. From $\bar{g}_Y = \bar{g}_C$, (22) and (47), the interest rate on the SGP in the corner solution case is determined as

$$\bar{r} = \frac{\Psi}{\alpha+\beta}(\underline{\delta}-\delta)n\sigma + \rho + n. \tag{95}$$

From (48), we can have $\bar{r} = \frac{\dot{v}}{v} + \frac{\pi}{v}$. Also, from (2), we have that the growth rate of π equals $\bar{g}_Y - \bar{g}_A$. With these conditions and the fact that the growth rates of π and v are same, we have

$$\frac{\pi}{v} = \bar{r} - \bar{g}_Y + \bar{g}_A. \tag{96}$$

We can solve above equation with (2), (5), $\bar{g}_A = B\{(1-u_{Y,c}^{sgp})\xi_c^{sgp}\}^{\phi}$, and $u_{Y,c}^{sgp} + u_{A,c}^{sgp} = 1$ as

$$u_{Y,c}^{sgp} = \frac{\Omega}{1+\Omega}, \quad \text{where} \quad \Omega = \beta\zeta(r - \bar{g}_Y + \bar{g}_A)/(\gamma(\zeta-\gamma)\bar{g}_A). \tag{97}$$

ξ_c^{sgp} follows immediately due to $\bar{g}_A = B\{(1-u_{Y,c}^{sgp})\xi_c^{sgp}\}^{\phi}$ as

$$\xi_c^{sgp} = \left(\frac{\bar{g}_A}{B}\right)^{\frac{1}{\phi}} \frac{1}{(1-u_{Y,c}^{sgp})}. \tag{98}$$

Next, from the resource constraint ($\dot{K} = Y - C$) and (2) we have

$$\frac{C}{K} = \frac{\bar{r}\zeta}{\gamma^2} - \bar{g}_Y = \frac{\tilde{c}_c^{sgp}}{\tilde{k}_c^{sgp}}. \tag{99}$$

Moreover, from $\frac{Y}{K} = \frac{\bar{r}\zeta}{\gamma^2}$ and (1) we have

$$\eta^{-\gamma} L^{\alpha} (u_{Y,c} H)^{\beta} A^{\zeta-\gamma} K^{\gamma-1} = \frac{\bar{r}\zeta}{\gamma^2}. \tag{100}$$

From the definition of $\tilde{k}_c$ and with a little algebra, we have

$$\tilde{k}_c^{sgp} = \left(\frac{\bar{r}\zeta\eta^{\gamma}}{\gamma^2 (u_{Y,c}^{sgp})^{\beta}}\right)^{\frac{-1}{1-\gamma}}. \tag{101}$$

Finally, with $\tilde{k}_c^{sgp}$, $\tilde{c}_c^{sgp}$ is obtained from (99) as

$$\tilde{c}_c^{sgp} = (\frac{\bar{r}\zeta}{\gamma^2} - \bar{g}_Y)\tilde{k}_c^{sgp}. \tag{102}$$

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Table 1: Key Economic Variables

Internal Solution Case						
$\delta = 0.9$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	−0.1564	0.0486	0.0868	0.1535	0.6092	0.2373
$\delta = 0.8$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	−0.0564	0.0592	0.0963	0.1762	0.5655	0.2584
Corner Solution Case						
$\delta = 0.9$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	−0.1564	0.0178	0.059	0.0527	0.9473	0
$\delta = 0.8$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	−0.0564	0.0256	0.066	0.098	0.902	0

Table 2: Eigen Values and the Number of Unstable Roots

$\delta = 0.9$		$\delta = 0.8$	
Eigen Values (I)	Eigen Values (C)	Eigen Values (I)	Eigen Values (C)
−4.36E+04	0.5834	−2.368E+04	−0.99921
0.4439	−0.9997	0.4876	0.53974
0.089277 + 0.023664i	0.0013	0.097501 + 0.018529i	0.0019168
0.089277 - 0.023664i	−8.00E-4	0.097501 - 0.018529i	−1.49E-03
− 8.18E-12		1.96E-09	
3	2	4	2
Saddle Stable	Saddle Stable	No Solution	Saddle Stable

The Number of Unstable Roots: (I) stands for the internal solution case and (C) stands for the corner solution case.

Table 3: Key Economic Variables

Internal Solution Case						
$\delta = 1$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	−0.1564	0.0381	0.0773	0.1245	0.6593	0.2162

Table 4: Eigen Values and the Number of Unstable Roots

$\delta = 1$
Eigen Values (I)
−9.13E+04
0.4004
0.080887 + 0.027064i
0.080887 - 0.027064i
− 3.18E-10
3
Saddle Stable

The Number of Unstable Roots: (I) stands for the internal solution case.