Endogenous Growth in the Absence of Instantaneous Market Clearing

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Abstract: Optimal growth models are designed to explain how the main economic aggregates evolve over time, from a given initial state to a long term steady state. In particular, endogenous growth models describe how a convergence (or divergence) process towards (away from) a constant growth scenario takes place. This process involves transitional dynamics, but typically there is a fundamental item that escapes dynamic adjustment: demand is, in every moment of time, equal to the output level, i.e., the goods market always clears. In the present paper, we develop an endogenous growth model in which market clearing is a long term possibility instead of an every period implicit assumption: the system may converge to a market equilibrium outcome in the same way it can converge to a state of constant growth. The implications of this modelling structure are essentially the following: a market clearing equilibrium may co-exist with other equilibrium points; several types of stability outcomes are possible to achieve; monetary policy becomes relevant to growth.

Keywords: Endogenous growth, Non-equilibrium models, Keynesian macroeconomics, Local stability analysis, Monetary policy.

JEL classification: O41, C62, E12.

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1. Introduction

The modern theory of economic growth studies long term wealth accumulation under the strong assumption that markets are permanently in an equilibrium position. The following sentence, by Cellarier (2006, p. 54), clearly states the assumption that is generally implicit in almost all the work involving the analysis of the growth process: ‘In equilibrium, total output along with the utilized aggregate level of capital and labor are all equal to their aggregate demand and supply (…). Therefore at any given time period, the market clearing real wage and real interest rate both depend on the current state of the economy’.

To assume that in each time period, from now to an undefined future, the demand level will be persistently equal to the quantity of produced goods can be interpreted as a somehow awkward hypothesis in the context of growth models. After all, noticing that growth setups are essentially intertemporal adjustment frameworks, we clearly understand that the notion of adjustment or convergence sharply contrasts with the conventional assumption that takes markets as automatically and instantaneously in equilibrium.

In this paper, we advocate that rather than being an automatic mechanism, market equilibrium is the result of a lengthy process that initiates in the present moment and that goes on to some point in the future, i.e., we consider the market equilibrium adjustment as an intertemporal phenomenon, just like the accumulation of material wealth. The main goal is to inquire about the consequences of approaching growth in simultaneous with market adjustment and to look at the long term results underlying such an approach. We will understand that the new assumption has a relevant impact over the growth process.

Our focus is on the growth implications of market disequilibria. We can associate this particular point of analysis to the more general debate on the overall role of the absence of instantaneous market clearing on the study of aggregate phenomena. What we know is that the velocity of market adjustment is far from being a neglected question in macroeconomics; on the contrary, it is in the core of the macroeconomic debate, as Mankiw (2006) points out. The classical tradition understands markets as mechanisms of automatic and instantaneous adjustment; the Keynesian view is one in which disequilibria tends to persist over time, given the inertial nature of markets, often subject to inefficiencies, information problems and coordination failures. It is not surprising, thus, that classical economics has always seemed more prepared to deal with long term growth, while Keynesian economics have focused in the explanation of short run cycles that are triggered by market frictions. The argument is that the analysis of growth is an analysis of long run trends, where one can neglect demand driven features and focus on the structural role of supply side economics. Typically, the classical view is one in which only accumulation of inputs matters for growth. All the discussion about frictions and the role of demand in shaping the market outcome is left for a conjuncture view about fluctuations (although even these may be looked at through the lenses of a classical perspective, as the Real Business Cycles theory does).

The 'new Keynesian perspective' [Clarida *et. al.* (1999)] or 'the new neoclassical synthesis' [Goodfriend and King (1997)], as different authors call the recent attempt to produce a consensus between classics and Keynesians in the macroeconomic analysis, mixes representative agent intertemporal optimization features with inherently Keynesian ideas, associated with nominal sluggishness. It begins to be consensual, in the contemporaneous analysis of macro phenomena, that the best ingredients of both perspectives should be combined, at the same time that the most counterfactual assumptions of the two views are purged, in order to obtain more robust explanations concerning economic reality. The weakest points in the Keynesian analysis have to do with the ad-hoc way in which some aggregate relations are established, while the classical view may be criticized by an excessively optimistic interpretation about market efficiency. From a classical point of view (as we said, the one underlying the main growth paradigms), the synthesis can be interpreted as a requirement to take seriously the factors that impose, in each moment, a lack of coincidence between aggregate demand and aggregate supply (being the main factor involved in this disequilibrium the impossibility of having, in the real world, completely flexible prices).

Besides the overall controversies, the disequilibrium or non-Walrasian approach to macroeconomics has constituted a relevant field of research on its own. Starting with the contributions of Patinkin (1965), Clower (1965), Leijonhufvud (1968), Barro and Grossman (1971), Bénassy (1975) and Malinvaud (1977), demand-supply imbalances and their consequences have been thoroughly debated. The implications of absence of market clearing in the goods market and / or in the labor market have to do essentially with the short run effects of excess demand or excess supply, with the eventual persistence of imbalances in time and with possible solutions to improve market efficiency, for instance, the discussion about the usefulness of a central auctioneer. The type of market structure that arises in face of markets that are not able to adjust automatically is also a matter of concern [e.g., Bénassy (1993, 2002) studies imbalances under a monopolistic competition environment].

Today's macroeconomics continues to address the issue of market disequilibrium in both product and labor markets. Short run economic implications of macro disequilibrium have been addressed, among others, by Flaschel *et. al.* (1997), Chiarella and Flaschel (2000), Asada *et. al.* (2003), Chiarella *et. al.* (2005), Raberto *et. al.* (2006) and Hallegatte and Ghil (2007). The referred authors use the disequilibrium approach to search for the presence of endogenous business cycles. They introduce market frictions, imperfect rationality in expectations and biases on aggregation as arguments to produce short run destabilizing effects that analytically translate in bifurcations capable of imposing a transition from a fixed point equilibrium to a region, in the parameters' space, where cycles of increasing periodicity and even completely a-periodic motion is observed.

What the literature tells us is that the study of 'Keynesian equilibria' [a term introduced by Geanakoplos and Polemarchakis (1986) to designate macroeconomic outcomes that deviate from the market clearing result] is, in fact, strongly associated with a short run vision of the economic system, a vision in which one may somehow relax the hypotheses of agent optimization, rational expectations and market clearing. Short term economic performance is more likely to be affected by publicly announced government policies and by less than rational behavior and expectations of private agents, in sharp contrast with the understanding one may take of long term trends of growth, which traditionally are viewed as solely determined by the supply side.

There are, nevertheless, good examples of attempts to deal with the long term impact of aggregate demand over growth trends. This is done, for instance, in Palley (1996, 2003), Blackburn (1999) and Dutt (2006). These authors begin precisely by highlighting that conventional macroeconomics only look to the interaction of aggregate demand and aggregate supply in the short run performance of the economy but it neglects the eventual role of demand on the analysis of economic growth: growth is exclusively driven by supply-side factors, like the state of technology. As Dutt (2006, p. 319) states: 'for mainstream macroeconomists, aggregate demand is relevant for the short run and in the study of cycles, but irrelevant for the study of growth'; then he asks (p. 320): 'is it not more sensible to have a growth theory in which both aggregate supply and aggregate demand considerations have roles to play?'

The main contribution provided by the mentioned authors concerns the eventual impact of aggregate demand over the steady state growth rate of the economy; their theoretical modelling structures point to the possibility of an increased rate of growth as the result of some demand shock, caused for instance by a change on the fiscal policy of the government. However, the introduction of Keynesian traits into growth setups does not have to change necessarily the long run growth results, typically dominated by supply side determinants. The fundamental issue relates to an exaggerated assumption of mainstream growth models, which relates to the fact that they simply ignore aggregate demand by assuming instantaneous market clearing. Aggregate demand can be irrelevant in the long run, but this does not automatically imply that its impact is absent on the transitional dynamic process. Furthermore, we may have long run supply driven results, but aggregate demand considerations may be determinant in shaping stability conditions that are decisive for knowing if the long term steady state is, in fact, achieved.

This paper departs from the previously mentioned contributions because it relates essentially to the analysis of transitional dynamics. It emphasizes the idea that an economy starting from a state of non balanced growth and non market equilibrium may converge to (or diverge from) a steady state of balanced growth and market clearing. Thus, the study of stability conditions is central to understand how the economy can attain the desirable long term rate of growth and also how this long term state may be compatible with market efficiency that is absent along the transitional dynamics phase. Therefore, we stress that determinants of long term growth are supply side entities, but the demand side is relevant to understand if such long run equilibrium is accomplishable and if it reflects an efficient allocation of resources.

In the sections that follow, we approach growth by introducing a disequilibrium mechanism in a classical endogenous growth model. We will consider mainly an AK model, as in Rebelo (1991), but the two-sector growth model with human capital, extensively discussed in the literature [e.g., Lucas (1988), Caballé and Santos (1993), Mulligan and Sala-i-Martin (1993), Bond, Wang and Yip (1996), Ladrón-de-Guevara, Ortigueira and Santos (1997), Gómez (2003, 2004)], is also developed.

The disequilibrium mechanism is essentially based on a pair of dynamic equations proposed in Hallegatte *et. al.* (2007) and it is able to eliminate instantaneous market clearing. However, as stated, market clearing is a long term possibility, i.e., in the same way the underlying system may converge to a stable constant growth rate, it can also converge to a market equilibrium result.

A one-sector, discrete-time and deterministic setup is presented and its local dynamics are addressed. The main variable of the reduced form system is a ratio between inventories and output; the stability of this ratio is discussed under an assumption of price stability (the central bank undertakes an optimal monetary policy aimed at a convergence to a positive, low and constant inflation rate). In this model, besides the disequilibrium mechanism, another Keynesian feature is present: a constant marginal propensity to consume, which turns consumption into a constant share of output.

Afterwards, we extend the model to include a human capital sector; in the two-sector model, we continue to assume absence of any optimization process (there is a constant marginal propensity to consume and the share of human capital in each of the two economic sectors is taken as constant over time and capable of allowing for an eventual market-clearing steady state). A second extension of the model puts it closer to the classical growth analysis by assuming consumption utility intertemporal maximization; the framework continues to assume the transitional dynamics market disequilibrium and the possibility of long run market equilibrium, but the representative agent chooses the level of consumption that maximizes intertemporal utility.

The remainder of the paper is organized as follows. Section 2 presents the basic structure of the model. Section 3 addresses the dynamic behavior of the system. Section 4 introduces the role of human capital by assuming a two-sector endogenous growth. In section 5 we take an optimal control problem of utility maximization, where the market non equilibrium mechanism is still present. Finally, section 6 is left to conclusions and implications.

2. The Disequilibrium Framework

Consider a closed economy, populated by a large number of agents (households and firms). In this economy, the population level does not grow and it coincides with the available labor; thus, by normalizing the amount of labor to 1, all variables to consider are simultaneously level variables and per capita / per unit of labor variables. Time is defined discretely, t=0,1,… and the developed setup will be fully deterministic.

In this economy, growth is endogenous. The adopted definition of endogenous growth is the following: all economic aggregates defined in real terms grow, in the steady state, at a constant rate $\gamma\in(-\delta,A-\delta)$. Parameter $\delta\geq 0$ is the depreciation rate of capital and $A>0$ translates the level of technology. The mentioned real aggregates are, for now, the following: $y_t\in\mathbb{R}_+$ (income / output); $k_t\in\mathbb{R}_+$ (physical capital); $d_t\in\mathbb{R}_+$

(demand); $c_t \in \mathbb{R}_+$ (consumption), $i_t \in \mathbb{R}_+$ (irreversible investment) and $h_t \in \mathbb{R}$ (goods inventory).

The endogenous growth nature of the problem derives from the assumption of an AK constant marginal returns production function, $y_t=Ak_t$. Capital accumulation, in turn, will correspond to the standard difference between investment and capital depreciation, as follows,

$$k_{t+1} - k_t = i_t - \delta k_t \text{, } k_0 \text{ given.} \tag{1}$$

The two equations that are essential to characterize the absence of market clearing are withdrawn from Hallegatte *et .al.* (2007) and they are the following,

$$h_{t+1} - (1+\gamma) \cdot h_t = y_t - d_t \text{, } h_0 \text{ given.} \tag{2}$$

$$p_{t+1} - p_t = -\theta \cdot p_t \cdot \frac{h_t}{d_t} \text{, } p_0 \text{ given.} \tag{3}$$

Equation (2) introduces the goods inventory variable. The meaning of this variable is understood by interpreting what different signs mean. If $h_t>0$, a situation of overproduction or a selling lag exists; it has correspondence on the time necessary for a firm to sell the produced goods. On the contrary, if $h_t<0$ then we have a case of underproduction or a delivery lag; it corresponds to the time needed for a consumer to get the ordered goods. Underproduction is the result of the technical lag associated with the transport and distribution of goods and to the inertia concerning changes in the installed production capacity. Equation (2) describes the way in which the lack of adjustment between production and demand changes the aggregate goods inventory. In the presence of market clearing (i.e., $y_t=d_t$), the goods inventory will grow at the benchmark growth rate γ; if output exceeds demand the goods inventory grows at a rate above γ, and if the output level is below the demand level the growth rate of the goods inventory is lower than γ.[1]

Equation (3) characterizes price changes as a function of the goods inventory per unit of demand (variable $p_t \in \mathbb{R}_+$ represents the price level). If inventories are positive,

[1] In Hallegatte *et. al.* (2007), equation (2) is just h_{t+1}-h_t=y_t-d_t. Their model is neoclassical (marginal returns are diminishing) and thus the economy does not grow in the long run. In this case, inventories grow positively for $y_t>d_t$, negatively for $y_t<d_t$ and they remain unchanged if $y_t=d_t$.

part of the production is not being sold, what makes buyers to concentrate market power and, thus, prices will have a tendency to fall. In the opposite case, the temporary underproduction attributes market power to the supply side, which drives a rise in prices. Parameter $\theta>0$ measures the degree of sensitivity of prices to inventory changes.

Equation (3) may be rewritten as

$$\pi_t = -\theta \cdot \frac{h_t}{d_t} \quad (4)$$

We define $\pi_t \equiv (p_{t+1} - p_t)/p_t$ as the inflation rate.

In the assumed closed economy with no government intervention at the fiscal level, demand is just given by $d_t=c_t+i_t$. Relatively to consumption, we adopt a simple Keynesian consumption function, $c_t=by_t$, with $b\in(0,1)$ the marginal propensity to consume. Investment is, then, given by the difference between demand and consumption, i.e., $i_t = -\theta \cdot \frac{h_t}{\pi_t} - by_t$. Because we have assumed that investment is irreversible, the following inequality must hold: $-h_t \geq \frac{b}{\theta} \cdot \pi_t y_t$. This inequality gives the relevant information that the goods inventory must have the opposite sign of the inflation rate. Later, we will take the reasonable assumption that, in the long term, inflation is positive, even though it should be close to zero; thus, the local analysis to unveil focus on the case in which underproduction prevails in the steady state; the long term growth equilibrium implies that there will permanently exist some inertia that prevents consumers to have immediate access to all the goods they need or want. The delivery lag becomes a permanent feature of the endogenous long run growth scenario.[2]

The dynamics of the model will be addressed by taking the ratio inventories per unit of output, $\varphi_t \equiv h_t/y_t$. Because both variables grow at a same steady state rate, the steady state value of φ_t is constant. A new equation of motion is presentable by combining equations (1), (2) and (4),

[2] At this point, we must be clear about our notions. We define market clearing as the circumstance in which $y_t=d_t$. This definition does not imply necessarily a zero inventories level ($h_t=0$). According to equation (2), market clearing just implies $h_{t+1}=(1+\gamma)\cdot h_t$, that is, that the inventories variable grows at rate γ.

$$\varphi_{t+1} = \frac{1+(1+\gamma)\cdot\varphi_t + \theta\cdot\varphi_t/\pi_t}{1-\delta - A\cdot(b+\theta\cdot\varphi_t/\pi_t)} \qquad (5)$$

The disequilibrium endogenous growth model can be studied in the presence of equation (5) and a rule concerning the motion of the inflation rate. Through equations (3) or (4), we have passed the idea that aggregate price changes are determined by the level of inventories. In practice, a responsible monetary policy acts in order to maintain price stability through the manipulation of nominal interest rates. Therefore, the inflation rate can be withdrawn from a monetary policy problem, and the inventories per unit of demand should change accordingly.

We will obtain a law of motion for inflation from a standard new Keynesian monetary policy problem. This problem is developed in appendix. Considering that the central bank: (i) acts optimally; (ii) aims at stability (i.e., the inflation rate should converge to its steady state level), the monetary authority will set an interest rate rule that implies the following equation of motion for inflation (see derivation in appendix):

$$\pi_{t+1} = \varepsilon_1\cdot\pi_t + (1-\varepsilon_1)\cdot\pi^* \qquad (6)$$

In equation (6), π^* defines the inflation rate target that the central bank chooses and $|\varepsilon_1|<1$ is a combination of policy parameters. It is straightforward to observe that the steady state level of inflation corresponds to the target defined by the central bank, $\bar{\pi} = \pi^*$, and that this is a stable steady state, given the constraint on the value of ε_1. The stability of equation (6) implies that the local dynamic analysis of the system may concentrate on the stability of equation (5), given a level of inflation that in the long run will surely be equal to the target that the monetary authority defines.

3. Steady State and Stability Analysis

We will be concerned with a specific kind of steady state: the market-clearing steady state. This is defined as the long term outcome in which the economy (all the previously defined real variables) grows at the constant rate γ and in which $\bar{y} = \bar{d}$. If this last condition applies, the capital constraint in (1) may be used to reveal which is in fact the long term growth rate; the result is: $\gamma = (1-b)\cdot A - \delta$. We rewrite equation (5) as an equation capable of producing long term market clearing (i.e., we replace the growth rate by the found expression),

$$\varphi_{t+1} = \frac{1 + [1 + (1-b) \cdot A - \delta] \cdot \varphi_t + \theta \cdot \varphi_t / \pi_t}{1 - \delta - A \cdot (b + \theta \cdot \varphi_t / \pi_t)} \qquad (7)$$

By solving $\overline{\varphi} \equiv \varphi_{t+1} = \varphi_t$ and $\overline{\pi} \equiv \pi_{t+1} = \pi_t$, two solutions for the inventories – output steady state ratio are encountered: $\overline{\varphi}_1 = -\pi^* / \theta$ and $\overline{\varphi}_2 = -1/A$. The first allows for market clearing; the second implies market clearing in the particular circumstance when $A = \theta / \pi^*$ (in fact, this case just means that a unique steady state is found). If the second steady state corresponds to the long term state of the economy, the long run scenario will be one of excess demand ($\overline{y} < \overline{d}$) if $A < \theta / \pi^*$, or one of excess supply ($\overline{y} > \overline{d}$) if $A > \theta / \pi^*$. Thus, even though we assume that the representative agent aims at a market clearing equilibrium (that satisfies her needs both as a consumer and as a producer) this may not be attained in the long term if the system converges to the alternative steady state $\overline{\varphi}_2$.

Therefore, it becomes essential to the analysis to inquire in which conditions each one of the steady states is stable or unstable. Before proceeding with the stability analysis, the following remarks about the steady state should be kept in mind:

i) The steady state $\overline{\varphi}_2$ corresponds, always, to a negative goods inventory; the same is true for $\overline{\varphi}_1$ under the assumption of a positive inflation target;

ii) The steady state investment-capital ratio is: $\overline{i} / \overline{k} = (1-b) \cdot A$ under $\overline{\varphi}_1$, and $\overline{i} / \overline{k} = \theta / \pi^* - bA$ under $\overline{\varphi}_2$. Considering the market clearing equilibrium, 1-*b* becomes the marginal propensity to invest; for the other steady state, the investment-capital ratio is lower than in the market clearing scenario if there is excess supply ($A > \theta / \pi^*$) and the opposite in case of excess demand. Since we have considered irreversible investment, an additional constraint is derived: $A < \theta / (b\pi^*)$;

iii) If there is not market clearing, the economy will grow in the steady state at a rate different from γ. Note that, in the steady state, the growth rate of the goods inventory is: $\frac{\overline{h}_{t+1} - \overline{h}_t}{\overline{h}_t} = \gamma + 1/\overline{\varphi} - \overline{d} / \overline{h}$; in the absence of market clearing $\overline{\varphi} = \overline{\varphi}_2$ and the demand – inventories ratio may be taken from equation (4); thus, the growth rate of the aggregate goods inventory in the steady state is:

$\frac{\bar{h}_{t+1}-\bar{h}_t}{\bar{h}_t}=\gamma-A+\theta/\pi^*=\theta/\pi^*-bA-\delta$. Since $\bar{\varphi}$ is constant and $\bar{d}/\bar{h}$ is constant, output and demand (as well as consumption and investment) will grow at this rate in the non market clearing equilibrium. This growth rate is above γ if there exists excess demand and it stays below γ under excess supply (this does not mean, however, that a situation of excess demand is preferable to market equilibrium, because although the economy grows more, there is always a part of the household needs that remain unfulfilled given the relative scarcity of supply, i.e., produced goods have to be rationed);

iv) Excess demand and excess supply in the steady state are quantifiable. Note that $\bar{y}=A\cdot|\bar{h}|$ and $\bar{d}=(\theta/\pi^*)\cdot|\bar{h}|$. In the presence of excess demand ($A<\theta/\pi^*$), the traded quantity is $A\cdot|\bar{h}|$, and the excess demand is $(\theta/\pi^*-A)\cdot|\bar{h}|>0$; in the case of excess supply ($A>\theta/\pi^*$), only the demanded quantity is traded, $(\theta/\pi^*)\cdot|\bar{h}|$, and the amount of excess supply is $(A-\theta/\pi^*)\cdot|\bar{h}|>0$.

In what concerns the stability analysis, knowing in anticipation that inflation converges to its target value, independently of the real economic conditions, means that we can concentrate on the inventories – output ratio equation. The following derivative is computed:

$$\left.\frac{\partial\varphi_{t+1}}{\partial\varphi_t}\right|_{(\bar{\varphi},\pi^*)}=$$
$$\frac{[1+(1-b)\cdot A-\delta+\theta/\pi^*]\cdot[1-bA-\delta-A\cdot(\theta/\pi^*)\cdot\bar{\varphi}]+[1+(1+(1-b)\cdot A-\delta+\theta/\pi^*)\cdot\bar{\varphi}]\cdot A\cdot(\theta/\pi^*)}{[1-bA-\delta-A\cdot(\theta/\pi^*)\cdot\bar{\varphi}]^2}$$

For each one of the steady state points:

$$\left.\frac{\partial\varphi_{t+1}}{\partial\varphi_t}\right|_{(\bar{\varphi}_1,\pi^*)}=\frac{1-bA-\delta+\theta/\pi^*}{1+(1-b)\cdot A-\delta};\quad \left.\frac{\partial\varphi_{t+1}}{\partial\varphi_t}\right|_{(\bar{\varphi}_2,\pi^*)}=\frac{1+(1-b)\cdot A-\delta}{1-bA-\delta+\theta/\pi^*}.$$

The relation $\left.\frac{\partial\varphi_{t+1}}{\partial\varphi_t}\right|_{(\bar{\varphi}_1,\pi^*)}=1\Big/\left.\frac{\partial\varphi_{t+1}}{\partial\varphi_t}\right|_{(\bar{\varphi}_2,\pi^*)}$ implies that if $\bar{\varphi}_1$ is inside the unit circle, $\bar{\varphi}_2$ will be located outside the unit circle and vice-versa. Therefore, the conclusion is that when one of the steady state points is stable, the other one is necessarily unstable (and the opposite). The possible stability cases for $\pi^*>0$ are synthesized in table 1.

*** Table 1 ***

Table 1 indicates the cases in which the market clearing and the non market clearing steady states are attained, given some initial value of the goods inventory – output ratio. We confirm that despite the desirableness of obtaining a steady state market equilibrium, for some combinations of parameters the economy diverges from such equilibrium and in the direction of a disequilibrium stable situation. The convergence process is determined by only three parameters: the degree of sensitivity of prices to inventory changes, the inflation target and the technological level.

Figure 1 depicts two alternative cases; the panel on the left refers to a stable market clearing steady state; the panel on the right has, as stable steady state, the one that perpetuates the market imbalance. The graphics are drawn for some $\pi_t = \pi^* > 0$.

*** Figure 1 ***

In figure 1, the bold line gives the position of the function $\varphi_{t+1}=f(\varphi_t)$ in equation (7) for a constant inflation rate. The way in which it intersects the 45º line determines the type of dynamics; the steady state to the left (the first point of intersection between the two lines) is always the stable one. Note that if φ_0>-1/A in the first panel, or φ_0>-π^*/θ in the second panel, the system will diverge, being impossible to achieve the stable long term outcome.

4. A Two-Sector Growth Model - the Role of Human Capital

In this section, we sophisticate the non equilibrium model by introducing a second productive sector: the education sector. The framework is similar to the one proposed in the literature, but the distribution of human capital across sectors will not be modelled as the result of an optimal choice. This distribution will correspond to the one that allows for a market clearing equilibrium, that is, the share of human capital allocated to the production of goods will now play the role that the propensity to consume had in the last section.

Consider the following changes over the non equilibrium model of the previous section. The final goods production function is changed to include a second input, which is human capital, $k_t^h \in \mathbb{R}_+$. This production function will be of the Cobb-Douglas type, i.e., there are diminishing returns associated to each one of the inputs (physical and human capital) and constant returns to scale (the function is homogeneous of degree one); we define two parameters: $\alpha \in (0,1)$, which corresponds to the output – physical capital elasticity and $u \in (0,1)$, which is the share of human capital used in the production of final goods; the remainder 1-u, is the share of human capital used as an input in the production of additional human capital. The production function is $y_t = Ak_t^{\alpha} \cdot (uk_t^h)^{1-\alpha}$. As before, A>0 represents the technology capabilities available at the goods production sector.

The accumulation of human capital obeys the rule

$$k_{t+1}^h - k_t^h = g\left[(1-u)\cdot k_t^h\right] - \delta k_t^h \quad , k_0^h \text{ given.} \tag{8}$$

According to equation (8), the only input of the human capital production function is human capital; the endogenous growth nature of the model is guaranteed by the assumption of a linear production function (i.e., there are constant marginal returns on the accumulation of human capital). Let this function be $g\left[(1-u)\cdot k_t^h\right] = g_t = B\cdot(1-u)\cdot k_t^h$, with B>0 the technology index of the education sector and g_t the output of the production of human capital. To simplify the analysis, it is assumed that human capital depreciates at the same rate δ as physical capital. Besides (8), the dynamic problem in consideration is also composed by equations (1), (2), (4) and (6).

The main difference of the setup in this section relatively to the one-sector case is that although we continue to assume that the disequilibrium (i.e., the lack of instantaneous market clearing) is associated to the final goods sector, now we transfer the source of endogenous growth to a second sector – the education sector. Observe that it is straightforward to realize that human capital grows at a constant rate, independently of the time moment, $\frac{k_{t+1}^h - k_t^h}{k_t^h} = B\cdot(1-u) - \delta$. If one defines the steady state as in the last section (all real variables grow at a same constant rate in the steady state), the presented growth rate is also the growth rate of physical capital, final goods output, demand, consumption and investment. Furthermore, if one continues focused on the idea of a market clearing steady state, this implies that the referred growth rate is equal to the parameter in the goods inventory dynamic equation, i.e., $\gamma = B\cdot(1-u)-\delta$.

Consider the following ratios: $\phi_t \equiv h_t / g_t$ and $\omega_t \equiv k_t / k_t^h$. These allow to obtain the system of equations (9)-(10),

$$\phi_{t+1} = \frac{\frac{Au^{1-\alpha}}{B\cdot(1-u)}\cdot\omega_t^{\alpha} + \left[1+B\cdot(1-u)-\delta+\frac{\theta}{\pi_t}\right]\cdot\phi_t}{1+B\cdot(1-u)-\delta} \tag{9}$$

$$\omega_{t+1} = \frac{(1-\delta)\cdot\omega_t - \frac{\theta}{\pi_t}\cdot B\cdot(1-u)\cdot\phi_t - bAu^{1-\alpha}\cdot\omega_t^{\alpha}}{1+B\cdot(1-u)-\delta} \tag{10}$$

Note that we have already replaced, in (9) and (10), the growth rate parameter by the corresponding expression. From the previous two equations, one obtains a unique steady state value for each one of the endogenous quotients: $\overline{\omega} = \left[\frac{(1-b)\cdot A}{B\cdot(1-u)}\right]^{1/(1-\alpha)}\cdot u$ and $\overline{\phi} = -\left[\frac{A}{B\cdot(1-u)}\right]^{1/(1-\alpha)}\cdot(1-b)^{\alpha/(1-\alpha)}\cdot u\cdot\frac{\pi^*}{\theta}$. Thus, a unique steady state exists. Observe that $\overline{\varphi} = \overline{\phi}\cdot\frac{B\cdot(1-u)}{Au^{1-\alpha}\cdot\overline{\omega}^{\alpha}} = -\frac{\pi^*}{\theta}$, that is, the obtained steady state is the one allowing for market clearing, with market clearing implying the same long term goods inventory – output ratio as in the one sector model. The main difference relatively to the one sector model is that because the growth rate is not determined in the disequilibrium sector, the second steady state (which did not allow for market clearing) no longer exists and, thus, our unique concern becomes to inquire about the stability of this market clearing steady state.

The linearization of the system in the steady state vicinity yields:

$$\begin{bmatrix}\phi_{t+1}-\overline{\phi}\\ \omega_{t+1}-\overline{\omega}\end{bmatrix} = \begin{bmatrix} 1+\frac{\theta/\pi^*}{1+B\cdot(1-u)-\delta} & \frac{\alpha}{(1-b)\cdot[1+B\cdot(1-u)-\delta]} \\ -\frac{\theta}{\pi^*}\cdot\frac{B\cdot(1-u)}{1+B\cdot(1-u)-\delta} & \frac{(1-b)\cdot(1-\delta)-\alpha bB\cdot(1-u)}{(1-b)\cdot[1+B\cdot(1-u)-\delta]}\end{bmatrix}\cdot\begin{bmatrix}\phi_t-\overline{\phi}\\ \omega_t-\overline{\omega}\end{bmatrix} \tag{11}$$

The trace and determinant of the Jacobian matrix in (11) are, respectively,

$$Tr(J) = 1 + \frac{(1-b)\cdot(1-\delta+\theta/\pi^{*}) - \alpha bB\cdot(1-u)}{(1-b)\cdot\left[1+B\cdot(1-u)-\delta\right]};$$

$$Det(J) = \frac{(1-b)\cdot(1-\delta) - \alpha bB\cdot(1-u)}{(1-b)\cdot\left[1+B\cdot(1-u)-\delta\right]} + \frac{\theta}{\pi^{*}}\cdot\frac{1+\alpha B\cdot(1-u)-\delta}{\left[1+B\cdot(1-u)-\delta\right]^{2}}.$$

Trace and determinant expressions tell us that a positive ratio θ/π^{*} requires $Tr(J)-1 > Det(J)$, i.e., feasible long term results imply that stability (two eigenvalues inside the unit circle) will not exist, since the stability condition $1-Tr(J)+Det(J)>0$ is violated.

Replacing the θ/π^{*} ratio in the determinant expression by the corresponding value in terms of the trace, one obtains $Det(J) = \nu + \frac{1+\alpha B\cdot(1-u)-\delta}{1+B\cdot(1-u)-\delta}\cdot Tr(J)$, with ν a combination of parameters *b*, *B*, *u*, α and δ. The previous expression indicates the presence of a determinant – trace relation with a positive but lower than one slope. Thus, the dynamics of the system is given by a relation that starts immediately after the bifurcation point $Det(J) = Tr(J)-1$ and with a slope lower than the one of the bifurcation line. This is depicted in figure 2. This figure reveals that the system is saddle-path stable for a relatively low value of the ratio θ/π^{*}, undergoing then a bifurcation that leads to instability.

*** Figure 2 ***

The saddle-path stability condition is:
$0 < \frac{\theta}{\pi^{*}} < \frac{B\cdot(1-u)\cdot\left[1-(1-\alpha)\cdot b\right]\cdot\left[1+B\cdot(1-u)-\delta\right]}{(1-b)\cdot\left[1+\alpha B\cdot(1-u)-\delta\right]}$. If the upper bound of the double inequality is crossed, then we no longer have $Det(J)<1$ and the stability possibility vanishes.

The two-sector model, that involves the presence of two forms of capital and where no control variable is considered, has a unique market clearing steady state which, for relatively low values of θ/π^{*}, is saddle-path stable. Thus, convergence to the steady state is not guaranteed independently of the initial point. The combination between the initial quantities of physical and human capital must be such that it puts the system on the stable trajectory allowing for convergence to the steady state; otherwise, variables will depart from the long run equilibrium.

5. Getting Closer to the Classics: Non-Equilibrium Intertemporal Optimization

In the previous sections, one has assumed that the representative agent does not optimize consumption or the distribution of inputs across sectors in order to obtain the best feasible intertemporal utility of consumption. However, non optimization is not a pre-requisite of the non market clearing analysis. In this section, we develop a one sector model similar to the one in sections 2 and 3 where, nevertheless, the constant marginal propensity to consume is replaced by a process of intertemporal utility maximization.

Let the representative consumer maximize

$$U_0 = \sum_{t=0}^{+\infty} \left[\beta^t \cdot u(c_t) \right] \tag{12}$$

Parameter $\beta \in (0,1)$ is the discount factor and $u(c_t)$ is the instantaneous utility function. We consider a simple continuous and differentiable utility function with diminishing marginal consumption utility of the logarithmic form, $u(c_t)=\ln(c_t)$. The maximization of U_0 is subject to resource constraints (1) and (2) and to the inflation equations (4) and (6). The main distinctive feature relatively to the problem of section 2, is that now consumption is not a fixed proportion of income, but the result of an optimization process.

Besides consumption, it is assumed that the inflation rate is also a control variable of the problem. The private economy is, after all, the first responsible by determining price changes. However, recall that the proposed framework gives a determinant role to the monetary authority in what concerns price setting: it is chosen an interest rate rule that makes prices to converge to a specified target. Thus, the assumption of optimization of the inflation rate, in order to maximize utility, by the representative agent is relevant if the monetary authority sets its policy taking into account the best interest of the private economy. In practical terms, the analytical treatment of the model will show that the possibility of control of inflation will not impose any constraint on the action of the central bank; it just sets a relation between shadow-prices capable of simplifying the problem's dynamics.

The Hamiltonian function of the problem is:

$$H(y_t, h_t, c_t, \pi_t, p_t^y, p_t^h) = u(c_t) - \beta p_{t+1}^y \cdot \left[A \cdot \left(\frac{\theta}{\pi_t} \cdot h_t + c_t \right) + \delta y_t \right]$$
$$+ \beta p_{t+1}^h \cdot \left[y_t + \left(\gamma + \frac{\theta}{\pi_t} \right) \cdot h_t \right]$$

$p_t^y, p_t^h \in \mathbb{R}$ are the co-state variables of y_t and h_t, respectively.

First-order optimality conditions are:

$$H_u = 0 \Rightarrow 1/c_t = A\beta p_{t+1}^y;$$

$$H_\pi = 0 \Rightarrow Ap_{t+1}^y = p_{t+1}^h;$$

$$\beta p_{t+1}^y - p_t^y = -H_y \Rightarrow (1-\delta) \cdot \beta p_{t+1}^y - p_t^y = -\beta p_{t+1}^h;$$

$$\beta p_{t+1}^h - p_t^h = -H_h \Rightarrow \left(1 + \gamma + \theta / \pi^*\right) \cdot \beta p_{t+1}^h - p_t^h = A \cdot (\theta / \pi^*) \cdot \beta p_{t+1}^y;$$

$$\lim_{t \to +\infty} y_t \beta^t p_t^y = 0 \text{ (transversality condition);}$$

$$\lim_{t \to +\infty} h_t \beta^t p_t^h = 0 \text{ (transversality condition).}$$

From the first three optimality conditions, it is obtained a constant growth rate for consumption:

$$\frac{c_{t+1} - c_t}{c_t} = \beta \cdot (1 + A - \delta) - 1 \tag{13}$$

Note that, as we have remarked, the second optimality condition does not impose any constraint that the central bank must obey when acting over the inflation rate; it just generates a relation between co-state variables such that consumption will grow in every period at a same rate.

The dynamic analysis will proceed by recovering the goods inventory – output ratio of section 2, φ_t, and by defining the consumption – output ratio $\psi_t \equiv c_t / y_t$. Once again, the steady state is defined as the state in which all real variables grow at a same constant rate and, thus, $\overline{\varphi}$ and $\overline{\psi}$ are constant values. A pair of difference equations describes the dynamics of the problem under study,

$$\varphi_{t+1} = \frac{1 + (1+\gamma) \cdot \varphi_t + \theta \cdot \varphi_t / \pi_t}{1 - \delta - A \cdot (\theta \cdot \varphi_t / \pi_t + \psi_t)} \tag{14}$$

$$\psi_{t+1} = \frac{\beta \cdot (1 + A - \delta)}{1 - \delta - A \cdot (\theta \cdot \varphi_t / \pi_t + \psi_t)} \cdot \psi_t \qquad (15)$$

The definition of steady state implies that the steady state growth rate must be the one presented in equation (13). The definition of market clearing steady state requires, as seen in section 3, that the long run growth rate is γ. Hence, for the case in appreciation, since one wants to discuss the possibility of convergence to market clearing, it is true that $\gamma = \beta \cdot (1 + A - \delta) - 1$. Replacing γ in system (14)-(15), we get a unique solution $\overline{\varphi} = -\pi^* / \theta$, $\overline{\psi} = \frac{1-\beta}{A} \cdot (1 + A - \delta)$. This steady state point allows for market clearing.

The stability of this equilibrium point is now addressed. In the vicinity of $(\overline{\varphi}, \overline{\psi})$, the system takes the linearized form:

$$\begin{bmatrix} \varphi_{t+1} - \overline{\varphi} \\ \psi_{t+1} - \overline{\psi} \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{\theta / \pi^* - A}{\beta \cdot (1 + A - \delta)} & -\dfrac{A\pi^* / \theta}{\beta \cdot (1 + A - \delta)} \\ \dfrac{1-\beta}{\beta} \cdot \dfrac{\theta}{\pi^*} & \dfrac{1}{\beta} \end{bmatrix} \cdot \begin{bmatrix} \varphi_t - \overline{\varphi} \\ \psi_t - \overline{\psi} \end{bmatrix} \qquad (16)$$

The trace and the determinant of the Jacobian matrix in (16) are, respectively, $Tr(J) = \frac{1+\beta}{\beta} + \frac{\theta / \pi^* - A}{\beta \cdot (1 + A - \delta)}$; $Det(J) = \frac{1}{\beta} + \frac{\theta / \pi^* - \beta A}{\beta^2 \cdot (1 + A - \delta)}$.

Stability conditions come,

$$1 + Tr(J) + Det(J) = 2 \cdot \frac{1+\beta}{\beta} + \frac{1}{\beta \cdot (1 + A - \delta)} \cdot \left(\frac{1+\beta}{\beta} \cdot \frac{\theta}{\pi^*} - 2A \right) > 0;$$

$$1 - Tr(J) + Det(J) = \frac{1-\beta}{\beta} \cdot \frac{1}{\beta \cdot (1 + A - \delta)} \cdot \frac{\theta}{\pi^*} > 0;$$

$$1 - Det(J) = -\frac{1-\beta}{\beta} - \frac{\theta / \pi^* - \beta A}{\beta^2 \cdot (1 + A - \delta)} > 0.$$

The second condition is satisfied for any positive π^* (we continue to consider that the inflation target is positive and, thus, the steady state goods inventory is a negative amount). The other two conditions may hold or may not hold, depending on parameter values. A diagram drawn in the trace-determinant referential will allow to understand under which conditions stability prevails.

For any positive θ/π^*, the relation between trace and determinant is $Det(J) = -\frac{1+\beta A-\delta}{\beta^2 \cdot (1+A-\delta)} + \frac{1}{\beta} \cdot Tr(J)$. The constraint $\theta/\pi^*>0$ requires $Tr(J) > 1 + \frac{1-\delta}{\beta \cdot (1+A-\delta)}$ and $Det(J) > \frac{1-\delta}{\beta \cdot (1+A-\delta)}$. Figure 3 presents a line that characterizes the possible stability outcomes.

*** Figure 3 ***

Only when above the bifurcation line $1 - Tr(J) + Det(J) = 0$ we have positive values of θ/π^*; thus, the system locates in this area. According to the graphic, the stability condition $1 + Tr(J) + Det(J) > 0$ holds and a possible bifurcation point exists for $Det(J) = 1$. The stability properties of the model are the following: if $(1-\beta)\cdot(1-\delta) < \beta A$, there is the possibility of existing a region of stability (the two eigenvalues of the matrix in (16) may lie inside the unit circle). This region of stability exists for $0 < \theta/\pi^* < \beta \cdot [\beta A - (1-\beta)\cdot(1-\delta)]$. The system is over a bifurcation point when $\theta/\pi^* = \beta \cdot [\beta A - (1-\beta)\cdot(1-\delta)]$ (a Neimark-Sacker bifurcation occurs at this point). Condition $\theta/\pi^* > \beta \cdot [\beta A - (1-\beta)\cdot(1-\delta)]$ implies instability.

Therefore, stability is synonymous of an inflation target rate above a given threshold $\pi^* > \theta / \{\beta \cdot [\beta A - (1-\beta)\cdot(1-\delta)]\}$. Recall that stability will imply two important long term achievements: utility maximization and market clearing.

A Numerical Example

To illustrate the obtained results, we consider a numerical example. Given the similarities in the dynamic process among the three presented growth problems (in the sense that they all allow for some kind of stability for an inflation target value above a given combination of parameter values), we restrict the application of the example to the optimization problem of this section. The values of the discount factor and of the depreciation rate are withdrawn from the calibration of a macro-model in Guo and Lansing (2002): β=0.962 and δ=0.067. The value of the parameter in equation (3) is the one in Hallegatte *et. al.* (2007), i.e., θ=0.0036. The technology index has to have a value that allows for a reasonable steady state growth rate; letting γ=0.05, then,

$A = (1+\gamma)/\beta - (1-\delta) = 0.1585$. Two additional parameter values are necessary, which are the ones in the monetary policy problem; here we adopt the values presented in table 6.4 (page 447) of Woodford (2003): a=0.048 and λ=0.024 (see the definition of these parameters in appendix).

The numerical example serves to show that, as long as stability holds, the transitional dynamics are characterized by a process of simultaneous convergence to a long term growth rate and to a long term market clearing result. The analytical treatment of the model implied a constraint on the value of the inflation rate target in order to guarantee stability. For the chosen parameter values, this constraint is π^*>0.032, that is, the monetary authority has to impose an inflation target above 3.2% if it wants the constant growth steady state to be accomplished. Otherwise, the model just diverges from equilibrium (in particular, the consumption – capital ratio falls to zero); if the system rests over the bifurcation line, i.e., if the inflation rate target is 3.2%, then perpetual cycles around the steady state will be evidenced.

The numerical example is graphically illustrated with figures 4 and 5, for an inflation target that guarantees stability (π^*=0.034). The figures correspond to the representation of the growth rate of output and of the demand-output ratio, respectively, from an initial point (φ_0,ψ_0,π_0)=(-9,0.5,0.05) till the observation 1,000.

*** Figures 4 and 5 ***

Figure 4 shows that the economy oscillates around a constant steady state growth rate of 5% and that it tends to it in the long run. Along with the growth process, there is a process of market convergence that leads the system in the direction of a long term market clearing state, where $\overline{d} = \overline{y}$, as revealed in figure 5; the system oscillates around the market clearing result but it will rest over that state only in the long term. The process of convergence will be as faster as the larger is the value of the inflation rate target.

Finally, we present figure 6, that respects to the behavior of ratio φ_t. In the present example, the goods inventory is systematically negative and it oscillates, with cycles of decreasing periodicity, towards the long run steady state $\overline{\varphi} = -\pi^* / \theta = -9.444$.

*** Figure 6 ***

6. Policy Implications and Discussion

Mainstream growth theory, including endogenous growth models, has always involved a time paradox; while the process of resource accumulation and generation of wealth is subject to an evolution from an initial state to a steady state, no similar process is found in what concerns market adjustment. Market equilibrium is instantaneous and no place is left for a dynamic transition from an initial state of excess demand or excess supply to a market clearing outcome.

The possibility of market adjustment is easily introduced by considered a goods inventory that is filled with increased output and emptied with increased demand. Establishing, then, a relation between the goods inventory and the growth of the price level, the framework of simultaneous growth and market disequilibrium becomes ready to the dynamic analysis. The behavior of the inflation rate can be understood, following the observed reality in modern economies, as the strict result of a monetary policy that uses the nominal interest rate as an instrument to directly determine the evolution of the price level.

The non equilibrium endogenous growth model was analyzed under three different settings. First, a simple consumption function, in which consumption is a constant share of the income level, was considered. This allowed addressing dynamics under a one-dimensional difference equation. Assuming that the representative agent aims at a market clearing steady state, one observes that there are two steady state points; one of them guarantees long term market clearing alongside with endogenous growth at a given rate, while the other leads to situations of excess demand or excess supply.

The study of stability indicates that the two points represent different stability outcomes (when one is stable, the other is unstable); in particular, the market clearing steady state prevails when the inflation rate target set by the central bank is bounded in a given interval; the boundaries of this interval are dependent on the technology index of the production function, on the depreciation rate of capital, on the marginal propensity to consume and on the elasticity between inflation and the goods inventory per unit of demand. In this way, we understand the relevant role of monetary policy over growth, which is absent in conventional growth models; the central bank may change the inflation target in order to guarantee a situation of long term market equilibrium (a situation where long term growth is compatible with a coincidence of interests between demand and supply agents).

On a second stage, an education sector was introduced. This has changed significantly the dynamics of the system because while the disequilibrium continues associated with the final goods sector, the source of endogenous growth is now linked to the linear shape of the human capital production function. A unique steady state exists when the representative agent searches for a market clearing equilibrium. The stability of this steady state point requires two important types of policy measures. First, the monetary authority should set the inflation rate target above a combination of parameters that involve the inflation – inventory sensitivity parameter, the capital depreciation rate, the marginal propensity to consume, the output – physical capital elasticity, the share of human capital allocated to the production of physical goods and the education sector technology (but not the goods sector technology). This relatively high value of the inflation target guarantees the existence of a saddle-path steady state.

The second policy measure has to do with the need of putting the system in the stable trajectory in order to converge to the long run equilibrium. Because the system does not involve any control variable, the private economy cannot produce this coincidence between an initial state and the convergence path. It has to be artificially generated by means of, for instance, a fiscal policy that drives the incentives for the accumulation of inputs. Basically, policy should be aimed at the formation of a relation between the amount of physical capital and human capital that puts the system over the saddle path.

Finally, a third model has abandoned the simple linear consumption function, replacing it by an optimization behavior of the private economy representative agent. This controls, as a household, the level of consumption and, as a firm, the way prices evolve over time. With these two control variables, the agent maximizes consumption utility. Once again, we have looked at the eventual existence of a market clearing steady state. It exists and it is unique. The analysis of local dynamics allows to find two possible stability outcomes: instability, for relatively low values of the inflation rate target (below a combination of parameters involving the technology of production, the parameter of the inflation – goods inventory per unit of demand equation, the depreciation of capital and the rate of discount of future utility), and stability (that exists regardless of the initial state) for relatively high values of the inflation rate target. Here, as in the other models, a low inflation rate target is advantageous because it allows for a not too negative steady state level of the goods inventory (i.e., the delivery lag is straightened if price increases are kept low); nevertheless, it can lead to an unstable outcome and therefore a divergence from the market clearing endogenous growth steady state.

The conclusion is that if one departs from the strong assumption of instantaneous market clearing when analyzing growth processes, monetary policy becomes relevant since price changes are an important influence over the real economy. Monetary policy should be such that the selected interest rate rule allows for a convergence to an inflation target. In turn, the inflation target must be low (to avoid cases of too high instantaneous underproduction) but not so low that it prevents convergence to the desired long run state.

A final point of interest concerns the comparison of the proposed setup with the basic AK growth model with instantaneous market clearing; this has plain and straightforward transitional dynamics, i.e., output grows in every time moment at exactly the same rate and the study of stability becomes irrelevant. In this sense, the assumption of a market adjustment process alongside with the growth process introduces the possibility of appealing dynamics otherwise absent and in which, as seen in the figures of the numerical example of section 5, periods of excess demand alternate with periods of excess supply, producing sequential phases of low growth and high growth (relatively to the long run rate).

Appendix – Derivation of the Optimal and Stable Interest Rate Rule

The goal of this appendix is to derive the inflation equation of motion (6) from a standard new Keynesian monetary policy model, presented as a fully deterministic problem in discrete time.

Consider that the monetary authority maximizes the value of function V_0,

$$V_0 = -\frac{1}{2} \cdot E_0 \left\{ \sum_{t=0}^{+\infty} \beta^t \cdot \left[a \cdot (x_t - x^*)^2 + (\pi_t - \pi^*)^2 \right] \right\} \qquad (a1)$$

Variables $\pi_t, x_t \in \mathbb{R}$ represent the inflation rate and the output gap; this last variable is defined as the difference, in logs, between effective and potential output. Parameters π^* and x^* are the target values for each one of the variables; these values are selected by the central bank according to its policy goals. The constant $a \geq 0$ represents the weight attributed to the real stabilization of the economy in the objective function, and $\beta \in (0,1)$ is the discount factor.

The two constraints of the problem are the usual IS dynamic equation (*a*2) and a new Keynesian Phillips curve, (*a*3),

$$x_t = -\zeta \cdot (r_t - E_t \pi_{t+1}) + E_t x_{t+1}, \ x_0 \text{ given.} \qquad (a2)$$

$$\pi_t = \lambda x_t + \beta \cdot E_t \pi_{t+1}, \ \pi_0 \text{ given.} \qquad (a3)$$

Terms $E_t\pi_{t+1}$ and $E_t x_{t+1}$ correspond to the private agents' expectations concerning next period inflation and output gap. Parameters $\zeta>0$ and $\lambda \in (0,1)$ are, respectively, the output gap – interest rate elasticity and a measure of price stickiness (the closer λ is from 0, the higher is the degree of price stickiness). Variable $r_t \in \mathbb{R}$, the nominal interest rate, is the control variable of the problem – the central bank will choose r_t in order to attain an optimal result, given the intertemporal objective function V_0.

Considering perfect foresight ($E_t\pi_{t+1}=\pi_{t+1}$ and $E_t x_{t+1}=x_{t+1}$), as one implicitly does in the analysis of the real economy across the text, and defining p_t and q_t as the co-state variables associated to x_t and π_t, respectively, one writes the Hamiltonian function of the problem,

$$\begin{aligned} H(x_t, \pi_t, r_t, p_t, q_t) = & -\frac{1}{2} \cdot \left[a \cdot (x_t - x^*)^2 + (\pi_t - \pi^*)^2 \right] \\ & + \beta p_{t+1} \cdot \left[-\zeta \cdot \left(r_t - \frac{1}{\beta} \cdot \pi_t + \frac{\lambda}{\beta} \cdot x_t \right) \right] \\ & + \beta q_{t+1} \cdot \left(\frac{1-\beta}{\beta} \cdot \pi_t - \frac{\lambda}{\beta} \cdot x_t \right) \end{aligned}$$

The first-order conditions are,

$$H_r = 0 \Rightarrow p_t = 0;$$

$$\beta \cdot p_{t+1} - p_t = -H_x \Rightarrow q_{t+1} = -\frac{a}{\lambda} \cdot (x_t - x^*);$$

$$\beta \cdot q_{t+1} - q_t = -H_\pi \Rightarrow q_{t+1} - q_t = \pi_t - \pi^*;$$

$$\lim_{t \to +\infty} x_t \cdot \beta^t \cdot p_t = 0 \ \text{(transversality condition);}$$

$$\lim_{t \to +\infty} \pi_t \cdot \beta^t \cdot q_t = 0 \ \text{(transversality condition).}$$

From the optimality conditions, one computes the following difference equation,

$$x_{t+1} = \left(1 + \frac{\lambda^2}{a\beta}\right) \cdot x_t - \frac{\lambda}{a\beta} \cdot \pi_t + \frac{\lambda}{a} \cdot \pi^* \qquad (a4)$$

Equation (*a*4) and the Phillips curve constitute a two-equation – two-endogenous variables linear system, which is presentable as,

$$\begin{bmatrix} x_{t+1} - \bar{x} \\ \pi_{t+1} - \bar{\pi} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\lambda^2}{a\beta} & -\frac{\lambda}{a\beta} \\ -\frac{\lambda}{\beta} & \frac{1}{\beta} \end{bmatrix} \cdot \begin{bmatrix} x_t - \bar{x} \\ \pi_t - \bar{\pi} \end{bmatrix} \qquad (a5)$$

By imposing $\bar{x} \equiv x_{t+1} = x_t$ and $\bar{\pi} \equiv \pi_{t+1} = \pi_t$, one obtains the steady state pair $(\bar{x};\bar{\pi}) = \left(\frac{1-\beta}{\lambda} \cdot \pi^*; \pi^*\right)$. The eigenvalues of the Jacobian matrix in (*a*5) are $0<\varepsilon_1<1$ and $\varepsilon_2>1$,

$$\varepsilon_1, \varepsilon_2 = \frac{a \cdot (1+\beta) + \lambda^2}{2a\beta} \mp \sqrt{\left[\frac{a \cdot (1+\beta) + \lambda^2}{2a\beta}\right]^2 - \frac{1}{\beta}} \qquad (a6)$$

Because one of the eigenvalues locates inside the unit circle and the other does not, the system is characterized by a saddle-path stable equilibrium: there is one stable dimension in the two-dimensional space that defines the system. Therefore, we can compute the saddle-path, i.e., the stable trajectory. This is $\pi_t - \bar{\pi} = \frac{p_2}{p_1} \cdot (x_t - \bar{x})$, with p_1 and p_2 the elements of an eigenvector associated to the eigenvalue inside the unit circle. One finds the expression $x_t = \frac{\beta \cdot (1-\varepsilon_1)}{\lambda} \cdot \pi^* - \frac{1-\beta\varepsilon_1}{\lambda} \cdot \pi_t$. Replacing the output gap expression just found into the Phillips curve, we obtain the inflation equation in (6).

We have obtained equation (6) in two steps:

i) First, the central bank builds an objective function and it adopts an optimizing behavior;

ii) Optimization leads to an infinite number of trajectories in the two dimensional space of the system, but only one of these trajectories is stable. Thus, the central bank also chooses to put the economy in the stable path, which guarantees the convergence to the inflation steady state value (that coincides with the target defined by the central bank).

The two previous goals are attained through the manipulation of the monetary policy instrument: the nominal interest rate. Note that from the IS equation (*a*2), the interest rate corresponds to the expression

$$r_t = \frac{1}{\zeta} \cdot (x_{t+1} - x_t) + \pi_{t+1} \qquad (a7)$$

Hence, imposing the interest rate rule in (*a*8), one obtains the inflation dynamic equation that simultaneously translates a situation of intertemporal optimization and selection of an inflation stable path,

$$r_t = \frac{1}{\zeta} \cdot (x_{t+1} - x_t) + \varepsilon_1 \cdot \pi_t + (1 - \varepsilon_1) \cdot \pi^* \qquad (a8)$$

Optimality and stability require the central bank to increase the nominal interest rate whenever there is an expected positive change in the output gap and whenever contemporaneous inflation rises (the opposite changes imply an interest rate decrease). Observe that under rule (*a*8), the steady state real interest rate is zero: $r^* = \pi^*$.

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Tables and Figures

Cases	Stable steady state	Market clearing
$A > \frac{2\cdot(1-\delta)}{2b-1}$ $\frac{\theta}{A} < \pi^* < \frac{\theta}{(2b-1)\cdot A - 2\cdot(1-\delta)}$	$\overline{\varphi}_1$	Yes
$A > \frac{2\cdot(1-\delta)}{2b-1}$ $0 < \pi^* < \frac{\theta}{A} \vee$ $\pi^* > \frac{\theta}{(2b-1)\cdot A - 2\cdot(1-\delta)}$	$\overline{\varphi}_2$	No $\begin{cases} \overline{d} > \overline{y} \text{ if } \pi^* < \frac{\theta}{A} \\ \overline{d} < \overline{y} \text{ if } \pi^* > \frac{\theta}{(2b-1)\cdot A - 2\cdot(1-\delta)} \end{cases}$
$A < \frac{2\cdot(1-\delta)}{2b-1}$ $\pi^* > \frac{\theta}{A}$	$\overline{\varphi}_1$	Yes
$A < \frac{2\cdot(1-\delta)}{2b-1}$ $0 < \pi^* < \frac{\theta}{A}$	$\overline{\varphi}_2$	No $\overline{d} > \overline{y}$

Table 1 – Steady state stability.

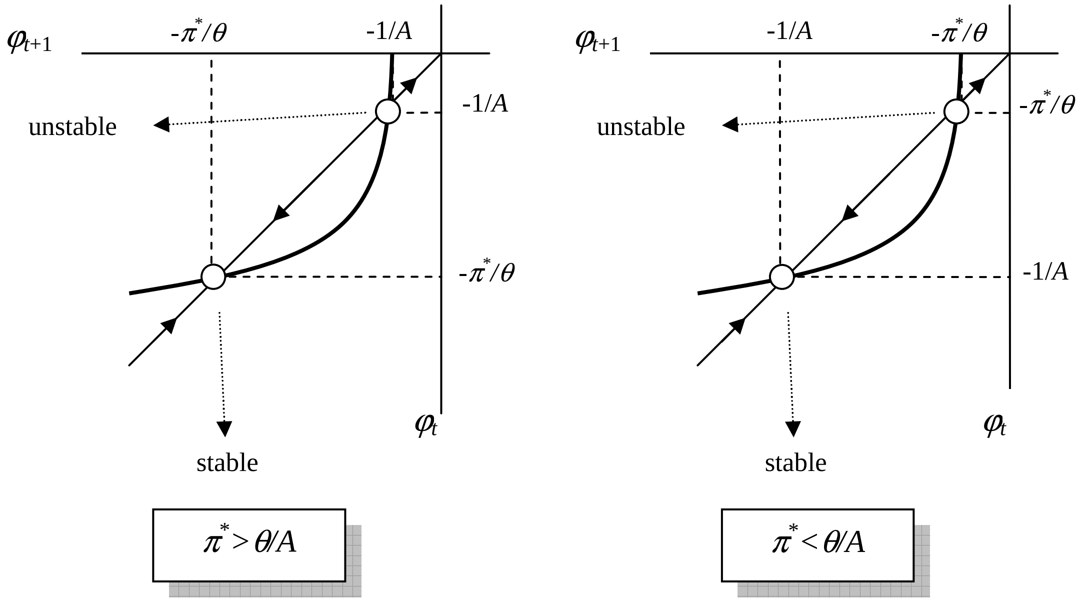

Figure 1 – Stability results in the constant propensity to consume one-sector model.

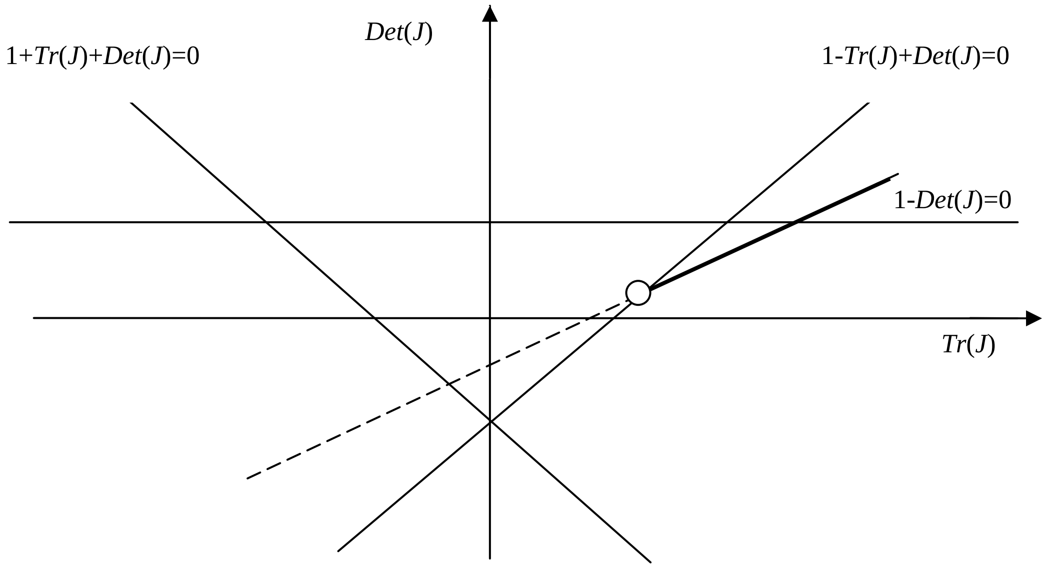

Figure 2 – Trace-determinant diagram in the two-sector model.

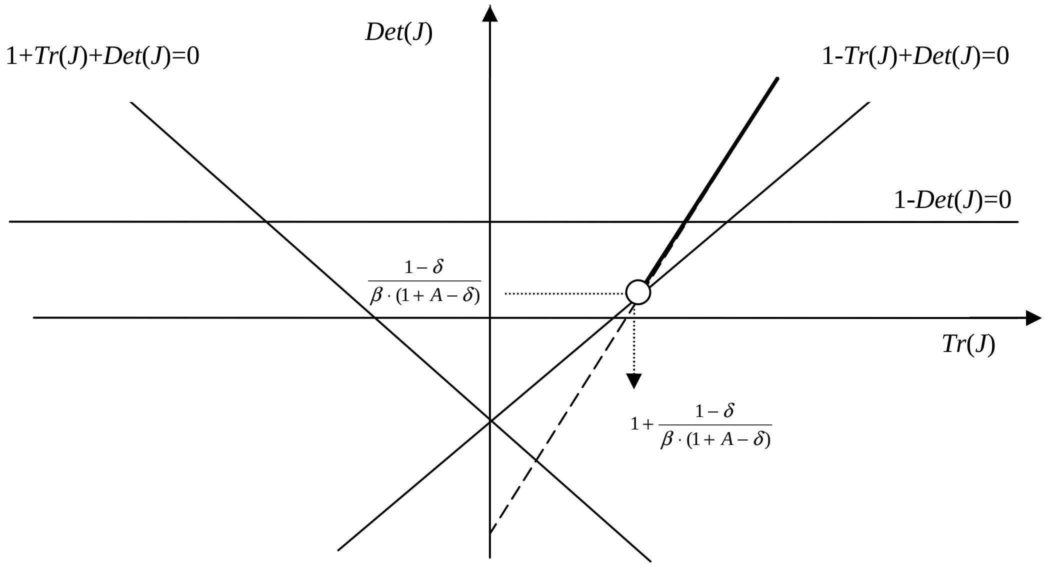

Figure 3 – Trace-determinant diagram in the optimization problem.

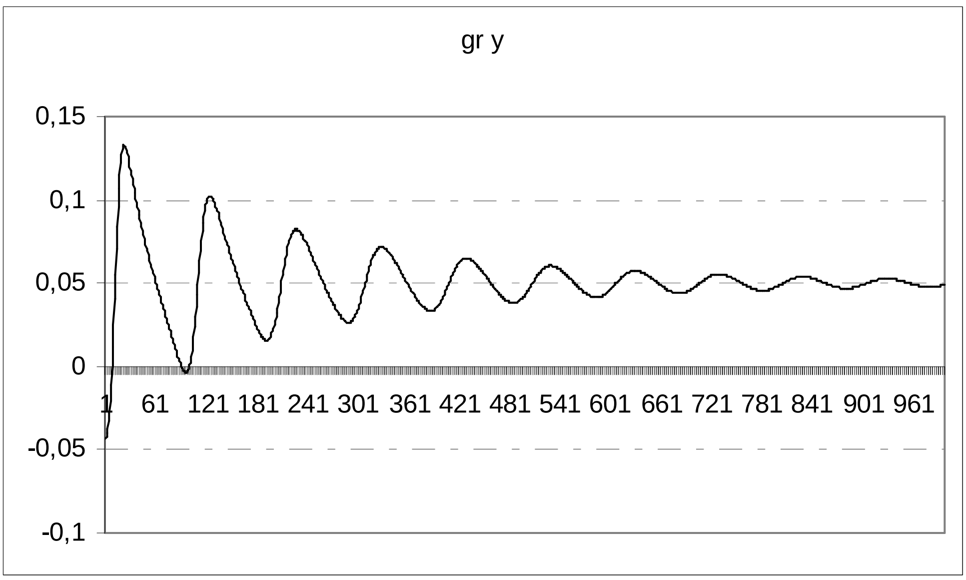

Figure 4 – Numerical example: output growth rate dynamics.

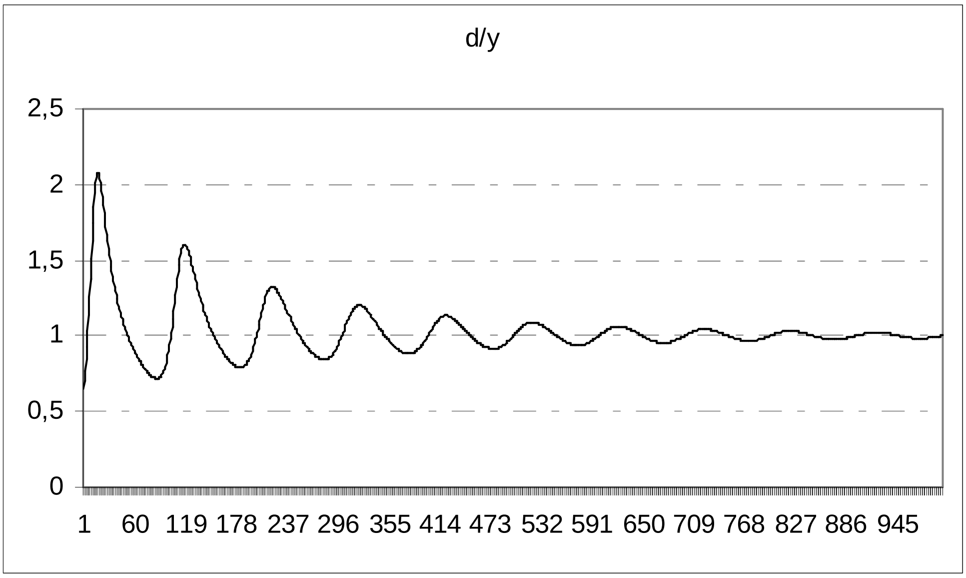

Figure 5 – Numerical example: demand-output ratio dynamics.

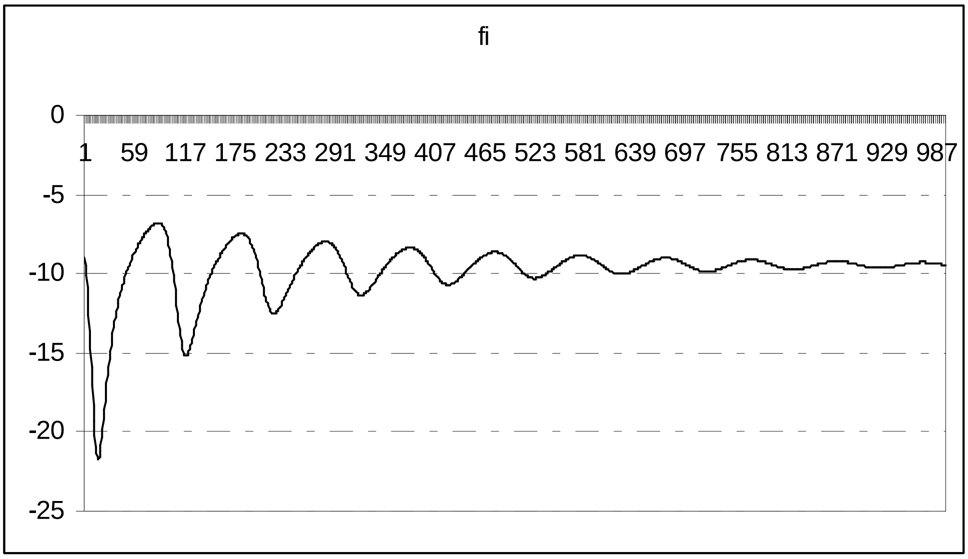

Figure 6 – Numerical example: goods inventory-output ratio dynamics.